Spring 2005 Math 152
7 Applications of Integration
7.1 Areas Between Curves
Fri, 21/Jan ©2005, Art Belmonte

Summary

Let $f(x)$ and $g(x)$ be continuous on the interval $[a, b]$. Then the area between the graphs of f and g is given by

$$A = \int_a^b |f(x) - g(x)|\, dx.$$

We compute this integral by resolving the absolute value into $f(x) - g(x)$ or $g(x) - f(x)$ according to whether $f(x) \geq g(x)$ or $g(x) \geq f(x)$, respectively, on various subintervals of $[a, b]$, then splitting up the integral if needed. (In most problems such splitting is unnecessary.)

Similarly, if $f(y)$ and $g(y)$ are continuous on the interval $[c, d]$, then the area between the graphs of f and g is given by

$$A = \int_c^d |f(y) - g(y)|\, dy$$

Analogous comments regarding splitting the integral apply.

Procedure for computing area

1. Draw a **sketch** of the curves involved and the region they enclose, preferably with MATLAB or a TI-89 graphing calculator; otherwise by hand. This is not required, but helps to visualize the situation.
2. Find the **intersections** of the two curves if necessary.
3. **Set up** the area integral.
4. **Compute** the value of this integral.

Hand Examples

For specificity, we assume lengths are in centimeters.

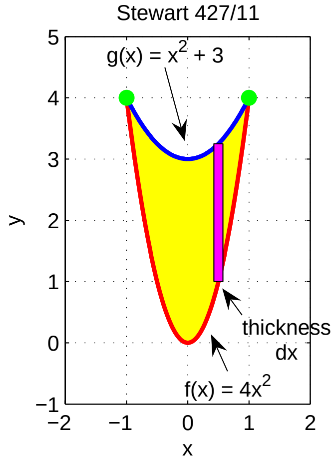
427/11

Find the area of the region bounded by the curves

$$y = f(x) = 4x^2 \text{ and } y = g(x) = x^2 + 3.$$

Solution

1. Here is a sketch of the curves and region.



2. When the curves intersect their y-coordinates are equal. Thus $4x^2 = x^2 + 3$, whence $x = \pm 1$ and $(x, y) = (\pm 1, 4)$.
3. There are only two intersections and $g(0) = 3 > 0 = f(0)$. We conclude that $g(x) - f(x) \geq 0$ on $[-1, 1]$. The area is given by $A = \int_a^b g(x) - f(x)\, dx$.

$$\int_{-1}^{1} \left(x^2 + 3\right) - \left(4x^2\right)\, dx = \int_{-1}^{1} 3 - 3x^2\, dx$$

4. Evaluate the integral. Verify with MATLAB or a TI-89.

$$\begin{aligned} \int_{-1}^{1} 3 - 3x^2\, dx &= \left(3x - x^3\right)\Big|_{-1}^{1} \\ &= (2) - (-2) = 4 \text{ cm}^2 \end{aligned}$$

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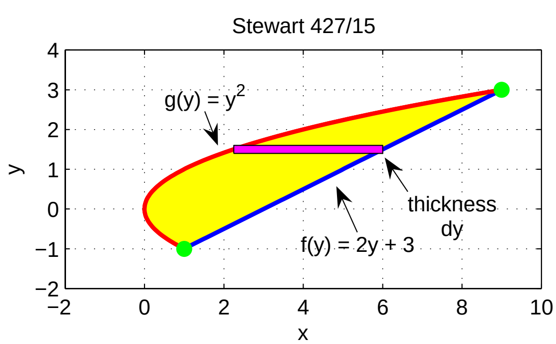
Find the area of the region bounded by the curves

$$x = f(y) = 2y + 3 \text{ and } x = g(y) = y^2.$$

Solution

1. Here is a sketch of the curves and region.



2. When the curves intersect their x-coordinates are equal. Thus $2y+3=y^2$, whence $y=-1,3$ and $(x,y)=(1,-1)$ or $(x,y)=(9,3)$.

3. There are only two intersections and $f(0)=3>0=g(0)$. We conclude that $f(y)-g(y)\geq 0$ on $[-1,3]$. The area is given by $A=\int_c^d f(y)-g(y)\,dy$.

$$\int_{-1}^{3}(2y+3)-\left(y^2\right)\,dy$$

4. Evaluate the integral. Verify with MATLAB or a TI-89.

$$\begin{aligned}\int_{-1}^{3}2y+3-y^2\,dy &= \left(y^2+3y-\tfrac{1}{3}y^3\right)\Big|_{-1}^{3}\\ &= (9)-\left(-\tfrac{5}{3}\right)=\tfrac{32}{3}=10\tfrac{2}{3}\\ &\approx 10.67\text{ cm}^2\end{aligned}$$

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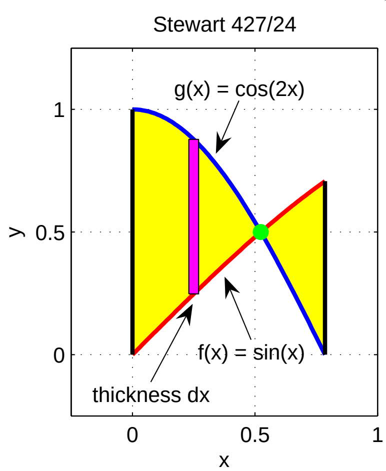
Find the area of the region bounded by the curves

$$y=f(x)=\sin x,\quad y=g(x)=\cos 2x,\quad x=0,\quad x=\tfrac{\pi}{4}.$$

Solution

1. Here is a sketch of the curves and region.



2. When the curves intersect their y-coordinates are equal. Let's find out where this occurs for $0\leq x\leq\frac{\pi}{4}$. We'll use the trig identity $\cos^2 x=\frac{1}{2}(1+\cos 2x)$.

$$\begin{aligned}\sin x &= \cos 2x\\ \sin x &= 2\cos^2 x-1\\ \sin x &= 2\left(1-\sin^2 x\right)-1\\ 2\sin^2 x+\sin x-1 &= 0\\ (2\sin x-1)(\sin x+1) &= 0\\ \sin x &= \tfrac{1}{2},-1\\ x &= \tfrac{\pi}{6},\ \text{since } 0\leq x\leq\tfrac{\pi}{4}.\end{aligned}$$

3. The area is given by $A=\int_a^b |f(x)-g(x)|\,dx$. From the plot we see that $\cos 2x\geq\sin x$ for $0\leq x\leq\frac{\pi}{6}$, whereas $\sin x\geq\cos 2x$ for $\frac{\pi}{6}\leq x\leq\frac{\pi}{4}$. Accordingly, we must split the integral into two pieces as follows.

$$A=\int_0^{\pi/6}\cos 2x-\sin x\,dx+\int_{\pi/6}^{\pi/4}\sin x-\cos 2x\,dx$$

4. Evaluate the integrals and add them. Verify with MATLAB or a TI-89.

$$\begin{aligned}&= \left(\tfrac{1}{2}\sin 2x+\cos x\right)\Big|_0^{\pi/6}+\left(-\cos x-\tfrac{1}{2}\sin 2x\right)\Big|_{\pi/6}^{\pi/4}\\ &= \left(\tfrac{3}{4}\sqrt{3}-1\right)+\left(-\tfrac{1}{2}\sqrt{2}-\tfrac{1}{2}-\left(-\tfrac{3}{4}\sqrt{3}\right)\right)\\ &= \tfrac{3}{2}\sqrt{3}-\tfrac{3}{2}-\tfrac{1}{2}\sqrt{2}\approx 0.39\text{ cm}^2\end{aligned}$$

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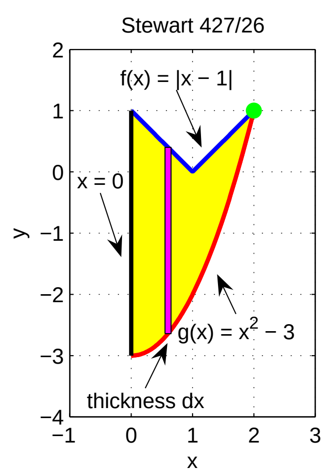
Find the area of the region bounded by the curves

$$y=f(x)=|x-1|\ \text{and}\ y=g(x)=x^2-3$$

along with the vertical line $x=0$.

Solution

1. Here is a sketch of the curves and region.



2. When the curves intersect their y-coordinates are equal. There are two cases.

   - If an intersection occurs at or to the right of $x=1$, then $|x-1|=x-1=x^2-3$, whence $x=-1,2$. But since $x\geq 1$, we have $x=2$ and thus $(x,y)=(2,1)$.
   - If an intersection occurs to the left of $x=1$, then $|x-1|=1-x=x^2-3$, whence $x=\dfrac{-1\pm\sqrt{17}}{2}\approx -2.56, 1.56$. Since $1.56\geq 1$, we toss it out. Moreover, since $-2.56<0$ (the left boundary of the region), we toss it out.

3. There is only one intersection in the region of interest and $f(0) = 1 > -3 = g(0)$. We conclude that $f(x) - g(x) \geq 0$ on [0, 2]. The area is given by $A = \int_a^b f(x) - g(x)\, dx$.

$$\int_0^1 (1-x) - \left(x^2 - 3\right) dx + \int_1^2 (x-1) - \left(x^2 - 3\right) dx$$

4. Evaluate the integrals and add them. Verify with MATLAB or a TI-89.

$$\begin{aligned} &= \int_0^1 4 - x - x^2\, dx + \int_1^2 x + 2 - x^2\, dx \\ &= \left(4x - \tfrac{1}{2}x^2 - \tfrac{1}{3}x^3\right)\Big|_0^1 + \left(\tfrac{1}{2}x^2 + 2x - \tfrac{1}{3}x^3\right)\Big|_1^2 \\ &= \left(\frac{19}{6} - 0\right) + \left(\frac{10}{3} - \frac{13}{6}\right) \\ &= \frac{19}{6} + \frac{7}{6} = \frac{26}{6} = \frac{13}{3} = 4\tfrac{1}{3} \approx 4.33 \text{ cm}^2 \end{aligned}$$

MATLAB Examples

From the hand examples you can see that there is a lot of work involved in applications of integration. Using MATLAB or a TI-89 to render graphics and perform symbolic computations is extremely helpful. Become proficient with technology to relieve the drudgery and increase your productivity!

s427x11 [427/11 revisited]

Find the area of the region bounded by the curves

$$y = f(x) = 4x^2 \text{ and } y = g(x) = x^2 + 3.$$

Solution

Here is a diary file that illustrates the many tasks that MATLAB can facilitate. (A TI-89 is also useful in this regard and provides portability.)

1. The x-coordinates of the points of intersection may be found with the **solve** command once x has been declared to be a symbolic variable with the **syms** command.
2. The **fill** command is used to draw the region of interest. One specifies the boundary of the region in a (counter)clockwise fashion. In this regard, the **fliplr** command flips the elements of a vector left-to-right. It sure beats reparameterizing a curve! Just concatenate the pieces of the boundary.
3. The **plot** command is used to draw the region's boundaries as well as to mark the points of intersection.
4. Various axis, label, and tick mark commands serve to make the figure prettier.
5. MATLAB's **int** command can symbolically compute the exact value of an integral. A TI-89 can do the same thing.
6. The **smi** command is one that I originally wrote for Calc 3 students to do multiple integration step-by-step. You can use it to do single integrals step-by-step and thus check your work! (Naturally, the same command is available in the TAMUCALC package that I wrote for the TI-89. This is available to you free-of-charge. See me if you are interested.)

```
%-----------------------------------------------
% Stewart 427/11
%
syms x
intersections = solve(4*x^2 - (x^2 + 3), x)
intersections =
  1
 -1
x = linspace(-1, 1);
f = 4*x.^2; g = x.^2 + 3;
xcc = [x fliplr(x)];
ycc = [f fliplr(g)];
%
fill(xcc, ycc, 'y', 'LineStyle', 'none')
grid on; hold on; axis equal
plot(x, f, 'r', 'LineWidth', 2)
plot(x, g, 'b', 'LineWidth', 2)
plot([-1 1], [4 4], 'go', 'MarkerFaceColor', 'g', ...
    'MarkerSize', 7)
%
axis([-2 2 -1 5])
xlabel('x'); ylabel('y')
title('Stewart 427/11')
set(gca, 'Xtick', -2:2)
set(gca, 'Ytick', -1:5)
% Slice
xs = 0.5; ys1 = 4*xs^2; ys2 = xs^2 + 3; h = 0.075;
fill([xs-h xs+h xs+h xs-h xs-h], ...
    [ys1 ys1 ys2 ys2 ys1], 'm')
%
syms x
A = int(3 - 3*x^2, x, -1, 1)
A =
4
A = smi(3 - 3*x^2, [x -1 1]);
STEPWISE (MULTIPLE) INTEGRATION!

Antiderivative w.r.t. x:

                                           3
                                    3 x - x
When x = 1:

                                          2
When x = -1:

                                         -2
Difference:

                                          4
Answer (above) and approximaton (below)
     4
%
echo off; diary off
```

s427x15 [427/15 revisited]

Find the area of the region bounded by the curves

$$x = f(y) = 2y + 3 \text{ and } x = g(y) = y^2.$$

Solution

The same principles used in s427x11 are employed in this problem. Note that y is the independent variable in this case.

```
%--------------------------------------------------
% Stewart 427/15
%
syms y
intersections = solve(2*y + 3 - y^2, y)
intersections =
 -1
  3
y = linspace(-1, 3);
f = 2.*y + 3; g = y.^2;
xcc = [f fliplr(g)];
ycc = [y fliplr(y)];
%
fill(xcc, ycc, 'y', 'LineStyle', 'none')
grid on; hold on; axis equal
plot(f, y, 'b', 'LineWidth', 2)
plot(g, y, 'r', 'LineWidth', 2)
plot([1 9], [-1 3], 'go', 'MarkerFaceColor', 'g', ...
    'MarkerSize', 7)
%
axis([-2 10 -2 4])
xlabel('x'); ylabel('y')
title('Stewart 427/15')
set(gca, 'Xtick', -2:2:10)
set(gca, 'Ytick', -2:4)
% Slice
ys = 1.5; xs1 = ys^2; xs2 = 2*ys + 3; h = 0.1;
fill([xs1 xs2 xs2 xs1 xs1], ...
    [ys-h ys-h ys+h ys+h ys-h], 'm')
%
syms y
A = int(2*y + 3 - y^2, y, -1, 3)
A =
32/3
A = smi(2*y + 3 - y^2, [y -1 3]);
STEPWISE (MULTIPLE) INTEGRATION!

Antiderivative w.r.t. y:

                              2              3
                             y  + 3 y - 1/3 y
When y = 3:

                                     9
When y = -1:

                                   -5/3
Difference:

                                   32/3
Answer (above) and approximaton (below)
   10.6667
%
echo off; diary off
```

s428x47

Find the approximate x-coordinates of the points of intersection of the curves $y = f(x) = \sqrt{x+1}$ and $y = g(x) = x^2$. The find the area of the region bounded by the curves (approximately).

Solution

In this problem, we employ some of the numerical routines for which MATLAB is renowned.

1. First off, we define f, g, and $h = f - g$ as separate function M-files. Make sure you use element-by-element operators (**.* ./ .^**) so that your functions can operate on vectors.
2. The **fzero** command finds where a function's value is zero. So when $h = f - g = 0$, we have $f = g$ and thus a point of intersection.
3. The **quad** command computes a single integral numerically. Since the x-coordinates of the points of intersection are approximate in this instance, computing the integral numerically makes sense.
4. Graphics are rendered in the same manner as before.

```
%--------------------------------------------------
% Stewart 428/47
%
a = fzero(@h, -0.5), b = fzero(@h,1)
a =
   -0.7245
b =
    1.2207
x = linspace(a, b);
yf = f(x); yg = g(x);
xcc = [x fliplr(x)];
ycc = [yg fliplr(yf)];
%
%
fill(xcc, ycc, 'y', 'LineStyle', 'none')
grid on; hold on; axis equal
plot(x, yf, 'b', 'LineWidth', 2)
plot(x, yg, 'r', 'LineWidth', 2)
plot([a b], [f(a) f(b)], 'go', 'MarkerFaceColor', 'g', ...
    'MarkerSize', 7)
%
axis([-1 2 -1 2])
xlabel('x'); ylabel('y')
title('Stewart 428/47')
set(gca, 'Xtick', -1:2)
set(gca, 'Ytick', -1:2)
% Slice
xs = 0.5; ys1 = xs^2; ys2 = sqrt(xs+1); hh = 0.04;
fill([xs-hh xs+hh xs+hh xs-hh xs-hh], ...
    [ys1 ys1 ys2 ys2 ys1], 'm')
%
A = quad(@h, a, b)
A =
    1.3767
%
echo off; diary off
%--------------------
function y = f(x)
y = sqrt(x+1);
%--------------------
function y = g(x)
y = x.^2;
%--------------------
function y = h(x)
y = f(x) - g(x);
```