Spring 2005 Math 152
7 Applications of Integration
7.4 Work
Mon, 31/Jan ©2005, Art Belmonte

Summary

- The work W done by a *constant* force F along a straight line through a distance d is $W = Fd$.
- Let $a < b$. The work W done by a *variable* force $F(x)$ along a straight line through a distance $b - a$ is $W = \int_a^b F(x)\,dx$.
- **Hooke's Law** from Physics states that the force $F(x)$ needed to maintain a spring stretched x units beyond its natural (unstretched) length is proportional to x. In other words, the force is $F(x) = kx$, where k is a constant of proportionality.

Hand Examples

448/2

How much work is done by a weightlifter in raising a 60 kg barbell from the floor to a height of 2 m?

Solution

The force involved is equal and opposite to the weight of the barbell. The mass of the barbell is $m = 60$ kg. Hence its weight is $F = mg$, where $g = 9.8\ \text{m/s}^2$. Accordingly, the work done is

$$W = Fd = mgd = (60\ \text{kg})\left(9.8\ \text{m/s}^2\right)(2\ \text{m}) = 1176\ \text{J}.$$

448/6

A spring has a natural length of 20 cm. If a 25 N force is required to keep it stetched to a length of 30 cm, how much work is required to stretch it from 20 cm to 25 cm?

Solution

- Determine the spring constant using Hooke's Law. Remember to convert centimeters to meters.

$$\begin{aligned} F(x) &= kx \\ 25 &= k\,(0.3 - 0.2) \\ k &= 250\ \text{N/m} \end{aligned}$$

  Therefore $F(x) = 250x$ N.

- Now compute the work. Recall that x is the amount the spring is stretched beyond its natural length (in meters).

$$\begin{aligned} W &= \int_a^b F(x)\,dx \\ &= \int_{0.2-0.2}^{0.25-0.2} 250x\,dx \\ &= 125x^2\Big|_0^{1/20} \\ &= \frac{5}{16} - 0 = 0.3125\ \text{J}. \end{aligned}$$

448/8

If the work required to stretch a spring 1 ft beyond its natural length is 12 ft-lb, how much work is needed to stetch it 9 in. beyond its natrual length?

Solution

- Determine the spring constant using Hooke's Law.

$$\begin{aligned} W &= \int_a^b F(x)\,dx \\ 12 &= \int_0^1 kx\,dx \\ 12 &= \tfrac{1}{2}kx^2\Big|_0^1 = \tfrac{1}{2}k \\ k &= 24 \end{aligned}$$

  Thus $F(x) = kx = 24x$.

- Now compute the work. Remember to convert inches to feet.

$$\begin{aligned} W &= \int_a^b F(x)\,dx \\ &= \int_0^{3/4} 24x\,dx \\ &= 12x^2\Big|_0^{3/4} \\ &= \frac{27}{4} - 0 = 6.75\ \text{ft-lb}. \end{aligned}$$

448/13

A cable whose weight density is $\delta = 2$ lb/ft is used to lift 800 lb of coal up a mineshaft 500 ft deep. Find the work done.

Solution

- Let x represent depth from the top of the mineshaft. Let's compute the amount of work dW needed to lift a differential slice dx of the rope to the top of the shaft.
  This slice weighs $\delta\, dx$ or $2\, dx$ lb. Hence
  $$dW = FD = (2\, dx)\, x = 2x\, dx$$
- Therefore, the work required to lift the rope is
  $$W = \int dW = \int_0^{500} 2x\, dx = x^2 \Big|_0^{500} = 250{,}000 \text{ ft-lb.}$$
- The work required to lift the coal is
  $$FD = (800)\,(500) = 400{,}000 \text{ ft-lb.}$$
- The *total* work is comprised of the work needed to lift the rope plus the work needed to lift the coal. This adds up to 650,000 ft-lb.

MATLAB Examples

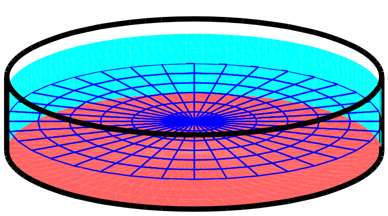
s448x16

A circular swimming pool has a diameter of 24 ft, the sides are 5 ft high, and the depth of the water is 4 ft. How much work is required to pump all of the water out over the side?

Solution

Here is a diagram of the swimming pool, the water within, and a differential layer of water. The density of water is $\delta = 62.5 \text{ lb/ft}^3$.

Stewart 448/16: Circular swimming pool



- The circular layer of water has a area of
  $$A = \pi r^2 = \pi\,(12)^2 = 144\pi.$$
  Its thickness is dz. Here are the volume of the layer, its weight, and the work required to lift it z ft.
  $$\begin{aligned} dV = A\,dz &= 144\pi\ dz \\ dF = \delta\, dV &= 144\delta\pi\ dz \\ dW = dF\, z &= 144\delta\pi z\, dz \end{aligned}$$
- The topmost layer of water must be lifted 1 ft over the top rim of the pool, whereas the layer at the bottom must be lifted 5 ft. Therefore, the work required to empty the swimming pool of water is
  $$W = \int dW = \int_1^5 144\delta\pi z\, dz = 72\delta\pi z^2 \Big|_1^5 = 1728\delta\pi$$
  In other words, the work is $(1728)\,(62.5)\,\pi \approx 339{,}292$ ft-lb.

```
%
syms delta z
W = int(144 * delta * pi * z,  z,  1,  5)
W =
1728*delta*pi
Work = round( subs(W, delta, 62.5) )
Work =
      339292
%
echo off; diary off
```

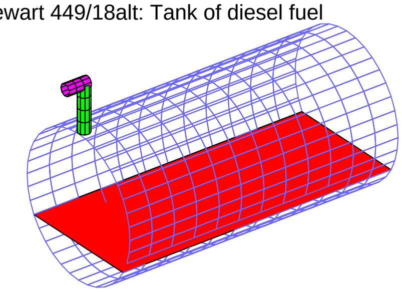
s449x18alt

A cylindrical tank of radius 1.5 m and length 6 m is filled with diesel fuel, the mass density of which is $\rho = 850 \text{ kg/m}^3$. The tank lies on its side. On the top of it is a spout whose outlet is 1 meter above the tank. Find the work required to pump all the diesel out through the outlet.

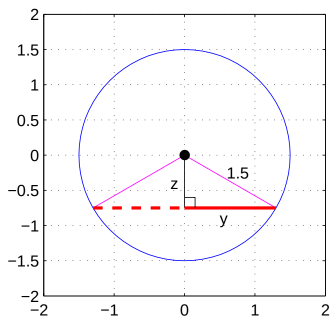
Solution

Here is a diagram of the tank and spout along with a rectangular differential layer of fuel.

Stewart 449/18alt: Tank of diesel fuel



Clearly the layer is 6 m long, but how wide is it? Let's look at a diagram of the circular end of the tank. The width of the layer is $2y = 2\sqrt{1.5^2 - z^2} = 2\sqrt{2.25 - z^2}$.



- The rectangular layer of water has a area of

$$A = LW = 6\left(2\sqrt{2.25 - z^2}\right) = 12\sqrt{2.25 - z^2}.$$

  Its thickness is dz. Here are the volume of the layer, its mass, its weight, and the work required to lift it to the top of the spout. Recall that $g = 9.8$ m/s^2.

$$\begin{aligned} dV = A\,dz &= 12\sqrt{2.25 - z^2}\,dz \\ dm = \rho\,dV &= 12\rho\sqrt{2.25 - z^2}\,dz \\ dF = (dm)\ g &= 12\rho g\sqrt{2.25 - z^2}\,dz \\ dW = dF\ (2.5 - z) &= 12\rho g\sqrt{2.25 - z^2}\,(2.5 - z)\ dz \end{aligned}$$

- The work required to pump the diesel out of the tank is

$$\begin{aligned} W &= \int dW \\ &= \int_{-1.5}^{1.5} 12\rho g\sqrt{2.25 - z^2}\,(2.5 - z)\ dz \\ &= 883{,}220 \text{ J.} \end{aligned}$$

```
%
% Stewart 449/18alts: Work done pumping diesel
%
syms delta g z
W = int(12 * delta * g * sqrt(2.25 - z^2) * (2.5 - z), ...
    z, -1.5, 1.5);
pretty(W)

                              135/4 pi delta g
Work = round( subs(W, {delta g}, {850 9.8}) )
Work =
      883220
%

echo off; diary off
```