Spring 2005 Math 152
7 Applications of Integration
7.5 Average Value of a Function
Wed, 02/Feb ©2005, Art Belmonte

Summary

- Let f be integrable on $[a,b]$. The **average value** of f over $[a,b]$ is defined by $f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)\,dx$.
- The **Mean Value Theorem for Integrals** states that for a function f continuous on $[a,b]$, we have
$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$$
for some $c \in [a,b]$. That is, f attains its average value somewhere in the interval $[a,b]$.

Hand Examples

451/6

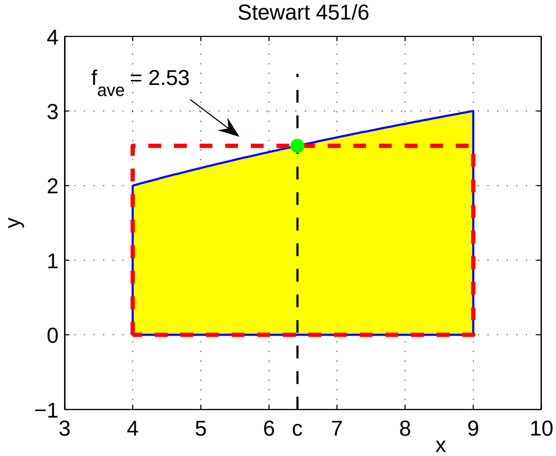
Find the average value of $f(x) = \sqrt{x}$ on the interval $[4,9]$, then illustrate the Mean Value Theorem for Integrals.

Solution

We have

$$\begin{aligned} f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)\,dx &= \frac{1}{9-4}\int_4^9 x^{1/2}\,dx \\ &= \left(\tfrac{1}{5}\right)\left(\tfrac{2}{3}\right)x^{3/2}\Big|_4^9 \\ &= \tfrac{2}{15}(27) - \tfrac{2}{15}(8) \\ &= \frac{2}{15}(19) = \frac{38}{15} \approx 2.53. \end{aligned}$$

The Mean Value Theorem for Integrals guarantees that f attains this average value somewhere in the interval. Indeed, we see that $f(c) = \sqrt{c} = \frac{38}{15}$ implies $c = \left(\frac{38}{15}\right)^2 \approx 6.42 \in [4,9]$. Here is a plot that summarizes these facts. Note that the area within the dashed rectangle is the same as the area of the shaded region.



451/10

Find the average value of $f(x) = x\sin(x^2)$ on $[0,\sqrt{\pi}]$, then illustrate the Mean Value Theorem for Integrals.

Solution

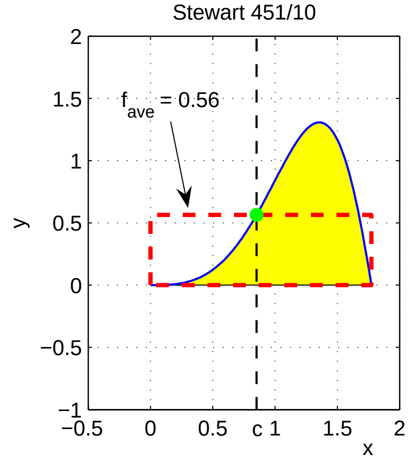
We have

$$f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{\sqrt{\pi}}\int_0^{\sqrt{\pi}} x\sin\left(x^2\right)\,dx = \star.$$

Use the Substitution Rule to compute the integral. Let $u = x^2$. Then $du = 2x\,dx$ or $\frac{1}{2}du = x\,dx$. Hence

$$\int_0^{\sqrt{\pi}} x\sin\left(x^2\right)\,dx = \frac{1}{2}\int_0^{\pi}\sin u\,du = \left(-\tfrac{1}{2}\cos u\right)\Big|_0^{\pi} = \tfrac{1}{2} - \left(-\tfrac{1}{2}\right) = 1.$$

Thus $f_{\text{ave}} = \star = 1/\sqrt{\pi} \approx 0.56$. The Mean Value Theorem for Integrals guarantees that f attains this average value somewhere in the interval. Indeed, we see that $f(c) = c\sin\left(c^2\right) = 1/\sqrt{\pi}$ implies $c \approx 0.85 \in \left[0,\sqrt{\pi}\right]$. Here is a plot that shows this. Note that the area within the dashed rectangle is the same as the area of the shaded region. [See MATLAB for computation of c.]



452/14

The temperature of a metal rod, 5 m long, is $f(x) = 4x$ (in °C) at a distance x meters from one end of the rod. What is the average temperature of the rod?

Solution

The average temperature is

$$\begin{aligned} f_{\text{ave}} &= \frac{1}{b-a}\int_a^b f(x)\,dx \\ &= \frac{1}{5-0}\int_0^5 4x\,dx \\ &= \tfrac{2}{5}x^2\Big|_0^5 \\ &= 10^\circ\text{C}. \end{aligned}$$

MATLAB Examples

s451x10

Here is the MATLAB code that produced the computations and figure for 451/10.

Solution

```
%
% Stewart 451/10: Average value & MVT for integrals
%
s = sqrt(pi)
s =
    1.7725
x = linspace(0, s); y = f(x);
fill(x, y, 'y'); hold on
plot(x, y, 'LineWidth', 1)
grid on; axis equal; axis([-0.5 2 -1 2])
%
f_ave = quad(@f, 0, s) / (s - 0)
f_ave =
    0.5642
c = fzero(@g, 1)
c =
    0.8512
plot([0 0 s s 0], [0 f_ave f_ave 0 0], ...
    'r--', 'LineWidth', 2)
xlabel('x'); ylabel('y')
title('Stewart 451/10')
plot([c c], [-1 2], 'k--', 'LineWidth', 1)
plot(c, f_ave, 'go', 'MarkerFaceColor', 'g')
text(c-0.035, -1.15, 'c')
%
echo off; diary off
%--------------------
function y = f(x)
y = x .* sin(x.^2);
%--------------------
function z = g(x)
z = f(x) - 1/sqrt(pi);
```

470/58

Household electricity is supplied in the form of alternating current that varies from 155 V to −155 V with a frequency of 60 cycles per second (Hertz or Hz). The voltage is given by the equation $E = E(t) = 155 \sin(120\pi t)$ where t is the time in seconds. Voltmeters read the RMS (root-mean-square) voltage, which is the square root of the average value of E^2 over one cycle. Find the RMS voltage of household current.

Solution

The argument of the sine function will vary from 0 to 2π as t varies from 0 to $\frac{1}{60}$. (Hence the phrase "60 cycles per second" or "60 Hertz.") Therefore,

$$\begin{aligned} V_{\text{RMS}} &= \sqrt{\frac{1}{b-a}\int_a^b E^2\,dt} \\ &= \sqrt{\frac{1}{\frac{1}{60}-0}\int_0^{1/60} (155\sin(120\pi t))^2\ dt} \\ &\approx 109.6 \text{ V.} \end{aligned}$$

```
%
% Stewart 470/58: RMS voltage of household current
%
syms t
f_ave = int( (155*sin(120*pi*t))^2, t, 0, 1/60) ...
    / (1/60 - 0)
f_ave =
24025/2
RMS = sqrt( eval(f_ave) )
RMS =
  109.6016
%
echo off; diary off
```