

# Spring 2005 Math 152

## 8 Techniques of Integration

### 8.1 Integration by Parts

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#### Summary

Integration by Parts:  $\int u dv = uv - \int v du.$

#### Hand Examples

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Evaluate  $\int x \ln x dx.$

#### Solution

Let  $u = \ln x$   $dv = x dx$   
 $du = \frac{1}{x} dx$   $v = \frac{1}{2}x^2$ . Then

$$\int x \ln x dx = \frac{1}{2}x^2 \ln x - \int \frac{1}{2}x dx = \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C.$$

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Evaluate  $\int \sin^{-1} x dx.$

#### Solution

Let  $u = \sin^{-1} x$   $dv = dx$   
 $du = \frac{1}{\sqrt{1-x^2}} dx$   $v = x$ . Then

$$\begin{aligned} \int \sin^{-1} x dx &= x \sin^{-1} x - \int (1-x^2)^{-1/2} x dx \\ &= x \sin^{-1} x + \sqrt{1-x^2} + C. \end{aligned}$$

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Evaluate  $\int t^3 e^t dt.$

#### Solution

Here we use integration by parts repeatedly.

- Let  $u = t^3$  and  $dv = e^t dt$ . Then  $du = 3t^2 dt$  and  $v = e^t$ .  
So  $\int t^3 e^t dt = t^3 e^t - \int 3t^2 e^t dt = t^3 e^t + \int -3t^2 e^t dt = \star$ .
- Let  $u = -3t^2$  and  $dv = e^t dt$ . Then  $du = -6t dt$  and  $v = e^t$ .  
Thus  $\star = t^3 e^t - 3t^2 e^t + \int 6t e^t dt = \star\star$ .
- Let  $u = 6t$  and  $dv = e^t dt$ . Then  $du = 6 dt$  and  $v = e^t$ .  
Hence  $\star\star = t^3 e^t - 3t^2 e^t + 6t e^t - \int 6e^t dt$ . Finish and factor to obtain  $(t^3 - 3t^2 + 6t - 6) e^t + C$ .

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Evaluate  $\int_0^{\pi/2} x \cos 2x dx.$

#### Solution

First compute an antiderivative, then apply the FTC.

- Let  $u = x$   $dv = \cos 2x dx$   
 $du = dx$   $v = \frac{1}{2} \sin 2x$ . Then

$$\int x \cos 2x dx = \frac{1}{2}x \sin 2x - \int \frac{1}{2} \sin 2x dx = \frac{1}{2}x \sin 2x + \frac{1}{4} \cos 2x.$$

- Hence  $\int_0^{\pi/2} x \cos 2x dx = \left( \frac{1}{2}x \sin 2x + \frac{1}{4} \cos 2x \right) \Big|_0^{\pi/2}$   
 $= \left( -\frac{1}{4} \right) - \left( \frac{1}{4} \right) = -\frac{1}{2}.$

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Evaluate  $\int \sin(\ln x) dx.$

#### Solution

Let  $u = \sin(\ln x)$  and  $dv = dx$ . Then  $du = \frac{\cos(\ln x)}{x} dx$  and  $v = x$ . Accordingly,

$$\int \sin(\ln x) dx = x \sin(\ln x) - \int \cos(\ln x) dx = \star.$$

Now let  $u = \cos(\ln x)$  and  $dv = -dx$ . Then  $du = -\frac{\sin(\ln x)}{x} dx$  and  $v = -x$ . Therefore,

$$\int \sin(\ln x) dx = \star = x \sin(\ln x) - x \cos(\ln x) - \int \sin(\ln x) dx$$

$$\text{whence } \int \sin(\ln x) dx = \frac{1}{2}x \left( \sin(\ln x) - \cos(\ln x) \right) + C.$$

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Evaluate  $\int \sin \sqrt{x} dx$  using the Substitution Rule followed by integration by parts.

#### Solution

Let  $w = x^{1/2}$ . Then  $dw = \frac{1}{2}x^{-1/2} dx$ , from which we conclude  $dx = 2\sqrt{x} dw = 2w dw$ . Hence  $\int \sin \sqrt{x} dx = \int 2w \sin w dw$ .

Let  $u = 2w$   $dv = \sin w dw$   
 $du = 2 dw$   $v = -\cos w$ . Then

$$\begin{aligned} \int 2w \sin w dw &= -2w \cos w + \int 2 \cos w dw \\ &= 2 \sin w - 2w \cos w + C \\ &= 2 \left( \sin \sqrt{x} - \sqrt{x} \cos \sqrt{x} \right) + C. \end{aligned}$$

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Use integration by parts to prove the reduction formula

$$\int x^n e^x dx = x^n e^x - n \int x^{n-1} e^x dx.$$

**Solution**

Let  $u = x^n$   $dv = e^x dx$   
 $du = nx^{n-1} dx$   $v = e^x$ . Then

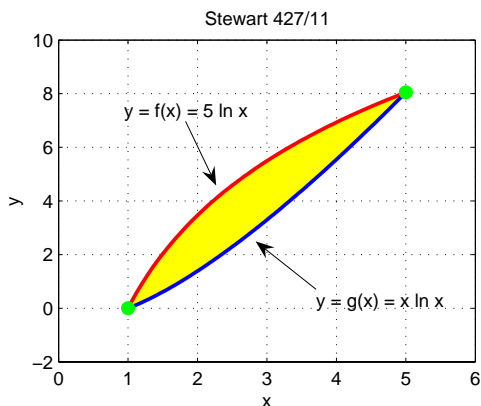
$$\int x^n e^x dx = x^n e^x - n \int x^{n-1} e^x dx.$$

464/48

Find the area of the region bounded by the curves  $y = 5 \ln x$  and  $y = x \ln x$ .

**Solution**

- When the curves intersect, we have  $5 \ln x = x \ln x$  or  $(5 - x) \ln x = 0$ . This implies  $x = 1, 5$ . Here is a plot.



- The area is  $A = \int_1^5 f(x) - g(x) dx = \int_1^5 (5 - x) \ln x dx$ . We need an antiderivative of the integrand.

- Let  $u = \ln x$   $dv = (5 - x) dx$   
 $du = \frac{1}{x} dx$   $v = 5x - \frac{1}{2}x^2$ . Then

$$\begin{aligned} \int (5 - x) \ln x dx &= \left(5x - \frac{1}{2}x^2\right) \ln x + \int \frac{1}{2}x - 5 dx \\ &= \left(5x - \frac{1}{2}x^2\right) \ln x + \frac{1}{4}x^2 - 5x. \end{aligned}$$

- Accordingly, the area is

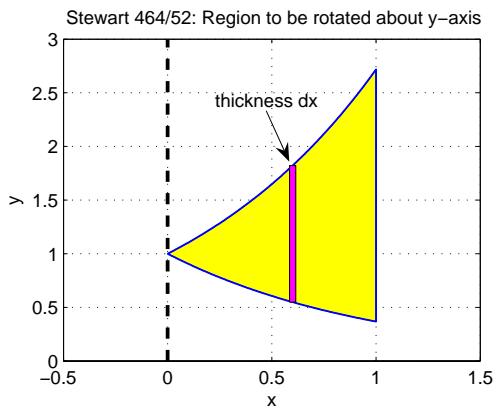
$$\begin{aligned} A &= \int_1^5 (5 - x) \ln x dx \\ &= \left( \left(5x - \frac{1}{2}x^2\right) \ln x + \frac{1}{4}x^2 - 5x \right) \Big|_1^5 \\ &= \left( \frac{25}{2} \ln 5 + \frac{25}{4} - 25 \right) - \left( \frac{1}{4} - 5 \right) \\ &= \frac{25}{2} \ln 5 - 14 \approx 6.12 \text{ cm}^2. \end{aligned}$$

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Use the method of cylindrical shells to find the volume generated by rotating the region bounded by  $y = e^x$ ,  $y = e^{-x}$ , and  $x = 1$  about the  $y$ -axis.

**Solution**

Here is a figure of the region to be rotated about the  $y$ -axis.



- The volume is

$$V = \int 2\pi rh dx = \int_0^1 2\pi x (e^x - e^{-x}) dx = 4\pi \int_0^1 x \sinh x dx.$$

We need an antiderivative of the integrand.

- Let  $u = x$   $dv = \sinh x dx$   
 $du = dx$   $v = \cosh x$ . Then

$$\begin{aligned} \int x \sinh x dx &= x \cosh x - \int \cosh x dx \\ &= x \cosh x - \sinh x. \end{aligned}$$

- Accordingly, the volume is

$$\begin{aligned} V &= 4\pi \int_0^1 x \sinh x dx \\ &= 4\pi (x \cosh x - \sinh x) \Big|_0^1 \\ &= 4\pi (\cosh 1 - \sinh 1) - 0 \\ &= 4\pi \left( \frac{e^1 + e^{-1}}{2} - \frac{e^1 - e^{-1}}{2} \right) = \frac{4\pi}{e} \approx 4.62 \text{ cm}^3. \end{aligned}$$

[NOTE: You may work directly with the exponential functions instead of using hyperbolic functions if you prefer.]

### MATLAB Examples

Recall that MATLAB can compute antiderivatives (indefinite integrals) or definite integrals via its **int** command. So can a TI-89 calculator via its **f(** command. Become familiar with machine power. You'll be glad you did!

Remember that **int** computes *an* antiderivative. You must mentally add the constant of integration. Moreover, the antiderivative it

gives may not look exactly like yours. But remember, it can only differ from yours by a constant.

Right; but can MATLAB and the TI-89 help us with integration by parts itself instead of just dumping out an answer? Sure thing! I wrote the **intparts** routine for just this purpose. The original version was for the TI-89. The MATLAB version is brand new and fancier. Here is the help for **intparts**, followed by some examples of the routine in action.

```
INTPARTS repeatedly performs integration by parts.
Given I (integrand), x (variable of integration),
and [optional] r (range of integration), intparts(I,x,r)
performs integration by parts at the direction of the
user, repeatedly if desired, in an interactive fashion.
At any time, the user may choose to have the routine
automatically finish the integration.
```

### s463x04

Evaluate  $\int x \ln x \, dx$ .

### Solution

Since we have an indefinite integral (antiderivative), we pass *two* arguments to **intparts**.

```
%-----
% Stewart 463/4s
%
syms x
MyAns = intparts(x*log(x), x);

INTEGRATION BY PARTS
-----

Cumulative antiderivative:

Current integrand:          0

u: Part to differentiate?
(or just ENTER to finish)

log(x)

          [u = log(x)   dv = x dx ]
          [             ]
          [   dx         ]
          [du = ----   v = 1/2 x ]
          [   x         ]

Press ENTER to continue.

INTEGRATION BY PARTS
-----

Cumulative antiderivative:

Current integrand:          2
          1/2 log(x) x

u: Part to differentiate?
(or just ENTER to finish)

          [u = 1   dv = - 1/2 x dx]
          [             ]
          [             ]
          [du = 0   v = - 1/4 x ]

Final antiderivative:

          2      2
          1/2 log(x) x - 1/4 x

%
echo off; diary off
```

### s463x12

Evaluate  $\int t^3 e^t \, dt$ .

### Solution

The **intparts** routine really shines when you have several instances of integration by parts!

```
%-----
% Stewart 463/12s
%
syms t
MyAns = intparts(t^3 * exp(t), t);

INTEGRATION BY PARTS
-----

Cumulative antiderivative:

Current integrand:          0

u: Part to differentiate?
(or just ENTER to finish)

t^3

          [ u = t^3   dv = exp(t) dt ]
          [             ]
          [ du = 3 t^2 dt   v = exp(t) ]

Press ENTER to continue.

INTEGRATION BY PARTS
-----

Cumulative antiderivative:

Current integrand:          2
          -3 exp(t) t

u: Part to differentiate?
(or just ENTER to finish)

-3*t^2

          [ u = -3 t^2   dv = exp(t) dt ]
          [             ]
          [ du = -6 t dt   v = exp(t) ]

Press ENTER to continue.

INTEGRATION BY PARTS
-----

Cumulative antiderivative:

Current integrand:          3
          t exp(t) - 3 exp(t) t

u: Part to differentiate?
(or just ENTER to finish)

6*t

          [ u = 6 t   dv = exp(t) dt ]
          [             ]
          [ du = 6 dt   v = exp(t) ]

Press ENTER to continue.

INTEGRATION BY PARTS
-----

Cumulative antiderivative:
```

```

          3          2
t exp(t) - 3 exp(t) t + 6 exp(t) t
Current integrand:

u: Part to differentiate?
(or just ENTER to finish)

          -6 exp(t)
          [ u = 1      dv = -6 exp(t) dt ]
          [           ]
          [ du = 0      v = -6 exp(t)  ]

Final antiderivative:

          3          2
t exp(t) - 3 exp(t) t + 6 exp(t) t - 6 exp(t)
pretty(factor(MyAns))

          3          2
exp(t) (t - 3 t + 6 t - 6)

%
echo off; diary off

```

### s463x19

Evaluate  $\int_0^{\pi/2} x \cos 2x \, dx$ .

### Solution

Since we have an definite integral, we pass *three* arguments to **intparts**.

```

%-----
% Stewart 463/19s
%
syms x
MyAns = intparts(x*cos(2*x), x, [0, pi/2]);

INTEGRATION BY PARTS
-----

Cumulative antiderivative:

Current integrand:

          0
          x cos(2 x)

u: Part to differentiate?
(or just ENTER to finish)

x

          [ u = x      dv = cos(2 x) dx ]
          [           ]
          [ du = dx     v = 1/2 sin(2 x) ]

Press ENTER to continue.

INTEGRATION BY PARTS
-----

Cumulative antiderivative:

Current integrand:

          1/2 x sin(2 x)

          - 1/2 sin(2 x)

u: Part to differentiate?
(or just ENTER to finish)

          [ u = 1      dv = - 1/2 sin(2 x) dx ]
          [           ]
          [ du = 0      v = 1/4 cos(2 x)  ]

Final antiderivative:

          1/2 x sin(2 x) + 1/4 cos(2 x)

When x = pi/2:

          -1/4

When x = 0:

          1/4

```

```

Difference:

Answer (above at center)
and approximation (below)
          -1/2
          -0.5000
%
echo off; diary off

```

### s464x48

Find the area of the region bounded by the curves  $y = 5 \ln x$  and  $y = x \ln x$ .

### Solution

See the corresponding hand example for the setup. Here is the code that performs the computations and renders the graphics. Here we just use **int** to compute the integral directly instead of having **intparts** walk us through the steps involved.

```

%-----
% Stewart 464/48
%
syms x
intersections = solve(5*log(x) - x*log(x), x)
intersections =
          5
          1
x = linspace(1, 5);
f = 5.*log(x); g = x .* log(x);
xcc = [x fliplr(x)];
ycc = [f fliplr(g)];
%
fill(xcc, ycc, 'y', 'LineStyle', 'none')
grid on; hold on;
plot(x, f, 'r', 'LineWidth', 2)
plot(x, g, 'b', 'LineWidth', 2)
plot([1 5], [0 5*log(5)], 'go', 'MarkerFaceColor', 'g', ...
      'MarkerSize', 7)
%
axis([0 6 -2 10])
xlabel('x'); ylabel('y')
title('Stewart 427/11')
set(gca, 'Xtick', 0:6)
set(gca, 'Ytick', -2:2:10)
%
syms x
A = int((5-x)*log(x), x, 1, 5); pretty(A)

          25/2 log(5) - 14

Area = eval(A)
Area =
          6.1180
%
echo off; diary off

```

### s464x48s2

Find the area of the region bounded by the curves  $y = 5 \ln x$  and  $y = x \ln x$ .

### Solution

This time we use **intparts** to walk us through the steps involved in the integration.

```

%-----
% Stewart 464/48s2
%
syms x
MyAns = intparts((5-x)*log(x), x, [1 5]);

```

INTEGRATION BY PARTS

-----

Cumulative antiderivative:

$$0$$

Current integrand:

$$(5 - x) \log(x)$$

u: Part to differentiate?  
(or just ENTER to finish)

log(x)

$$\begin{array}{l} [u = \log(x) \quad dv = 5 - x \, dx \quad ] \\ [ \quad \quad \quad \quad \quad \quad \quad ] \\ [ \quad \quad \quad dx \quad \quad \quad \quad 2] \\ [du = \frac{dx}{x} \quad v = 5x - 1/2 x^2 \quad ] \\ [ \quad \quad \quad \quad \quad \quad \quad ] \end{array}$$

Press ENTER to continue.

INTEGRATION BY PARTS

-----

Cumulative antiderivative:

$$\log(x) (5x - 1/2 x^2)$$

Current integrand:

$$\frac{-5x + 1/2 x^2}{x}$$

u: Part to differentiate?  
(or just ENTER to finish)

$$\begin{array}{l} [ \quad \quad \quad \quad \quad \quad \quad 2 \quad ] \\ [ \quad \quad \quad (-5x + 1/2 x^2) \, dx \quad ] \\ [u = 1 \quad dv = \frac{dx}{x} \quad ] \\ [ \quad \quad \quad \quad \quad \quad \quad ] \\ [ \quad \quad \quad \quad \quad \quad \quad ] \\ [du = 0 \quad v = -5x + 1/4 x^2 \quad ] \end{array}$$

Final antiderivative:

$$\log(x) (5x - 1/2 x^2) - 5x + 1/4 x^2$$

When x = 5:

$$25/2 \log(5) - 75/4$$

When x = 1:

$$-19/4$$

Difference:

$$25/2 \log(5) - 14$$

Answer (above at center)  
and approximation (below)  
6.1180

%  
echo off; diary off