Spring 2005 Math 152
8 Techniques of Integration
8.2 Trigonometric Integrals
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Summary

Recall and memorize the following trigonometric identities.

$$\begin{array}{ll} \sin^2 x + \cos^2 x = 1 & \sin^2 x = \frac{1}{2}(1 - \cos 2x) \\ \tan^2 x + 1 = \sec^2 x & \cos^2 x = \frac{1}{2}(1 + \cos 2x) \\ 1 + \cot^2 x = \csc^2 x & \sin 2x = 2 \sin x \cos x \end{array}$$

$$\sin A \cos B = \tfrac{1}{2}\big(\sin(A-B) + \sin(A+B)\big)$$
$$\sin A \sin B = \tfrac{1}{2}\big(\cos(A-B) - \cos(A+B)\big)$$
$$\cos A \cos B = \tfrac{1}{2}\big(\cos(A-B) + \cos(A+B)\big)$$

Also memorize these two bizarre formulas. The first is derived in the textbook, then second in 470/37 in the hand examples. Both use obscure tricks.

$$\begin{aligned} \int \sec x \, dx &= \ln|\sec x + \tan x| + C \\ \int \csc x \, dx &= \ln|\csc x - \cot x| + C \end{aligned}$$

Hand Examples

470/6

Evaluate $\int \sin^4 x \cos^3 x \, dx$.

Solution

Convert all but one of the *odd* number of cosine factors into sines.

$$\begin{aligned} \int \sin^4 x \cos^3 x \, dx &= \int \sin^4 x \left(1 - \sin^2 x\right) \cos x \, dx \\ &= \int \left(\sin^4 x - \sin^6 x\right) \cos x \, dx \\ &= \tfrac{1}{5} \sin^5 x - \tfrac{1}{7} \sin^7 x + C \end{aligned}$$

[We implicitly used the Substitution Rule. Let $w = \sin x$. Then $dw = \cos x \, dx$ and $\int w^4 - w^6 \, dw = \frac{1}{5} w^5 - \frac{1}{7} w^7 + C$.]

470/8

Evaluate $\int_0^{\pi/2} \sin^2 x \cos^2 x \, dx$.

Solution

Convert the squares of sine and cosine, then repeat.

$$\begin{aligned} \sin^2 x \cos^2 x &= \tfrac{1}{2}(1 - \cos 2x) \cdot \tfrac{1}{2}(1 + \cos 2x) \\ &= \tfrac{1}{4}\left(1 - \cos^2 2x\right) \\ &= \tfrac{1}{4}\left(1 - \tfrac{1}{2}(1 + \cos 4x)\right) \\ &= \tfrac{1}{8} - \tfrac{1}{8}\cos 4x = \tfrac{1}{8}(1 - \cos 4x) \end{aligned}$$

Therefore, $\int_0^{\pi/2} \sin^2 x \cos^2 x \, dx$ is equivalent to

$$\frac{1}{8} \int_0^{\pi/2} 1 - \cos 4x \, dx = \frac{1}{8}\left(x - \frac{1}{4}\sin 4x\right)\Bigg|_0^{\pi/2} = \frac{\pi}{16} \approx 0.20.$$

470/18

Evaluate $\int \frac{1}{1 - \sin x} \, dx$.

Solution

Multiply numerator and denominator by $1 + \sin x$.

$$\begin{aligned} \int \frac{1}{1 - \sin x} \, dx &= \int \frac{1 + \sin x}{1 - \sin^2 x} \, dx \\ &= \int \frac{1 + \sin x}{\cos^2 x} \, dx \\ &= \int \sec^2 x + \sec x \tan x \, dx \\ &= \tan x + \sec x + C \end{aligned}$$

470/24

Evaluate $\int_0^{\pi/4} \tan^2 x \sec^4 x \, dx$.

Solution

Convert all but two of the *even* number of secant factors into tangents.

$$\begin{aligned} \int_0^{\pi/4} \tan^2 x \sec^4 x \, dx &= \int_0^{\pi/4} \tan^2 x \left(\tan^2 x + 1\right) \sec^2 x \, dx \\ &= \int_0^{\pi/4} \left(\tan^4 x + \tan^2 x\right) \sec^2 x \, dx \\ &= \left(\tfrac{1}{5}\tan^5 x + \tfrac{1}{3}\tan^3 x\right)\Bigg|_0^{\pi/4} \\ &= \left(\frac{1}{5} + \frac{1}{3}\right) - 0 = \frac{8}{15} \approx 0.53 \end{aligned}$$

Again note the implicit use of the Substitution Rule in the penultimate step.

470/30

Evaluate $\int_0^{\pi/3} \tan^5 x \sec^3 x \, dx$.

Solution

Convert all but one of the *odd* number of tangent factors into secants. Season to taste.

$$\begin{aligned}
\int_0^{\pi/3} \tan^5 x \sec^3 x \, dx &= \int_0^{\pi/3} \left(\tan^2 x\right)^2 \sec^2 x \cdot \sec x \tan x \, dx \\
&= \int_0^{\pi/3} \left(\sec^2 x - 1\right)^2 \sec^2 x \cdot \sec x \tan x \, dx \\
&= \int_0^{\pi/3} \left(\sec^2 x - 2\sec^4 x + \sec^6 x\right) \cdot \sec x \tan x \, dx \\
&= \int_1^2 w^2 - 2w^4 + w^6 \, dw \\
&= \left(\tfrac{1}{3}w^3 - \tfrac{2}{5}w^5 + \tfrac{1}{7}w^7\right)\Big|_1^2 \\
&= \left(\frac{2^3}{3} - \frac{2^6}{5} + \frac{2^7}{7}\right) - \left(\frac{1}{3} - \frac{2}{5} + \frac{1}{7}\right) = \frac{848}{105} \approx 8.08
\end{aligned}$$

Here we explicitly used the Substitution Rule. Letting $w = \sec x$, we have $dw = \sec x \tan x \, dx$ and

x	0	$\frac{\pi}{3}$
w	1	2

.

470/32

Evaluate $Z = \int \tan^2 x \sec x \, dx$.

Solution

Let $\begin{array}{ll} u = \tan x & dv = \sec x \tan x \, dx \\ du = \sec^2 x \, dx & v = \sec x \end{array}$. Then

$$\begin{aligned}
Z = \int \tan^2 x \sec x \, dx &= \sec x \tan x - \int \sec^3 x \, dx \\
&= \sec x \tan x - \int \left(1 + \tan^2 x\right) \sec x \, dx \\
&= \sec x \tan x - \int \sec x \, dx - Z.
\end{aligned}$$

Hence $Z = \frac{1}{2}\left(\sec x \tan x - \int \sec x \, dx\right)$. This in turn is equal to

$$\tfrac{1}{2}\big(\sec x \tan x - \ln|\sec x + \tan x|\big) + C.$$

470/37

Derive the bizarre formula $\int \csc x \, dx = \ln|\csc x - \cot x| + C$.

Solution

Multiply "numerator" and "denominator" by $(\csc x - \cot x)$, an admittedly obscure trick!

$$\begin{aligned}
\int \frac{\csc x}{1} \, dx &= \int \frac{\csc x}{1} \cdot \frac{(\csc x - \cot x)}{(\csc x - \cot x)} \, dx \\
&= \int \frac{\csc^2 x - \csc x \cot x}{\csc x - \cot x} \, dx \\
&= \ln|\csc x - \cot x| + C
\end{aligned}$$

[We implicitly used the Substitution Rule with $w = \csc x - \cot x$ and $dw = \left(\csc^2 x - \csc x \cot x\right) dx$. Thus $\int \frac{1}{w} \, dw = \ln|w| + C$.]

470/41

Evaluate $\int \cos 3x \cos 4x \, dx$.

Solution

We have

$$\int \cos 3x \cos 4x \, dx = \tfrac{1}{2}\int \cos x + \cos 7x \, dx = \tfrac{1}{2}\left(\sin x + \tfrac{1}{7}\sin 7x\right) + C.$$

470/44

Evaluate $\int \frac{\cos x + \sin x}{\sin 2x} \, dx$.

Solution

Replace $\sin 2x$ by $2 \sin x \cos x$, then divide.

$$\begin{aligned}
\int \frac{\cos x + \sin x}{\sin 2x} \, dx &= \int \frac{\cos x + \sin x}{2 \sin x \cos x} \, dx \\
&= \frac{1}{2}\int \frac{1}{\sin x} + \frac{1}{\cos x} \, dx \\
&= \frac{1}{2}\int \csc x + \sec x \, dx \\
&= \frac{1}{2}\big(\ln|\csc x - \cot x| + \ln|\sec + \tan x|\big) + C
\end{aligned}$$

MATLAB Examples

Recall that MATLAB can compute antiderivatives (indefinite integrals) or definite integrals via its **int** command. So can a TI-89 calculator via its $\int($ command. Become familiar with machine power. You'll be glad you did!

Remember that **int** computes *an* antiderivative. You must mentally add the constant of integration. Moreover, the antiderivative it gives may not look exactly like yours. But remember, it can only differ from yours by a constant.

s470x06

Evaluate $\int \sin^4 x \cos^3 x \, dx$.

Solution

```
%------------------------------------------------
% Stewart 470/6
%
syms x
f = sin(x)^4 * cos(x)^3; pretty(f)

                                        4       3
                                  sin(x)  cos(x)
F = int(f, x);
pretty(F)

                3       4                       4
  - 1/7 sin(x)  cos(x)  - 3/35 sin(x) cos(x)
                 2
  + 1/35 cos(x)  sin(x)  +  2/35 sin(x)
%
ByHand = sin(x)^5 / 5  -  sin(x)^7 / 7;
pretty(ByHand)

                                      5             7
                           1/5 sin(x)  - 1/7 sin(x)
no_diff = simple(F - ByHand)
no_diff =
0
%
echo off; diary off
```

Despite the fact that machine and hand answers look different, they are equivalent in this instance since their difference is zero.

s470x08

Evaluate $\int_0^{\pi/2} \sin^2 x \cos^2 x \, dx$.

Solution

```
%------------------------------------------------
% Stewart 470/8
%
syms x
a = int(sin(x)^2 * cos(x)^2, x, 0, pi/2);
pretty(a)

                                    1/16 pi
a_float = eval(a)
a_float =
    0.1963
%
echo off; diary off
```

s470x18

Evaluate $\int \frac{1}{1 - \sin x} \, dx$.

Solution

```
%------------------------------------------------
% Stewart 470/18
%
syms x
f = 1 / (1 - sin(x)); pretty(f)

                                      1
                                  ----------
                                  1 - sin(x)
F = int(f, x);
pretty(F)

                                       2
                               - --------------
                                 tan(1/2 x) - 1
%
ByHand = tan(x) + sec(x);
pretty(ByHand)

                                tan(x) + sec(x)
const_diff = simple(F - ByHand)
const_diff =
1
%
echo off; diary off
```

While machine and hand answers are different, they are both antiderivatives of $\frac{1}{1 - \sin x}$ since their difference is 1, a constant.

s470x24

Evaluate $\int_0^{\pi/4} \tan^2 x \sec^4 x \, dx$.

Solution

```
%------------------------------------------------
% Stewart 470/24
%
syms x
a = int(tan(x)^2 * sec(x)^4, x, 0, pi/4);
pretty(a)

                                        8/15
a_float = eval(a)
a_float =
    0.5333
%
echo off; diary off
```

s470x30

Evaluate $\int_0^{\pi/3} \tan^5 x \sec^3 x \, dx$.

Solution

```
%------------------------------------------------
% Stewart 470/30
%
syms x
a = int(tan(x)^5 * sec(x)^3, x, 0, pi/3);
pretty(a)
%
```

```
                                           848
                                           ---
                                           105
a_float = eval(a)
a_float =
    8.0762
%
echo off; diary off
```

s470x32

Evaluate $\int \tan^2 x \sec x \, dx$.

Solution

```
%-------------------------------------------------
% Stewart 470/32
%
syms x
f = tan(x)^2 * sec(x); pretty(f)

                                 2
                           tan(x)  sec(x)
F = int(f, x);
pretty(F)

             3
       sin(x)
   1/2 ------- + 1/2 sin(x) - 1/2 log(sec(x) + tan(x))
             2
       cos(x)
%
ByHand = 1/2 * ( sec(x)*tan(x) - log(sec(x) + tan(x)) );
pretty(ByHand)

   1/2 sec(x) tan(x) - 1/2 log(sec(x) + tan(x))
no = simple(F - ByHand)
no =
0
%
echo off; diary off
```

Despite the fact that machine and hand answers look different, they are equivalent in this instance since their difference is zero.

s470x41

Evaluate $\int \cos 3x \cos 4x \, dx$.

Solution

```
%-------------------------------------------------
% Stewart 470/41
%
syms x
f = cos(3*x) * cos(4*x); pretty(f)

                              cos(3 x) cos(4 x)
F = int(f, x);
pretty(F)

                         1/2 sin(x) + 1/14 sin(7 x)
%
echo off; diary off
```

s470x44

Evaluate $\int \frac{\cos x + \sin x}{\sin 2x} \, dx$.

Solution

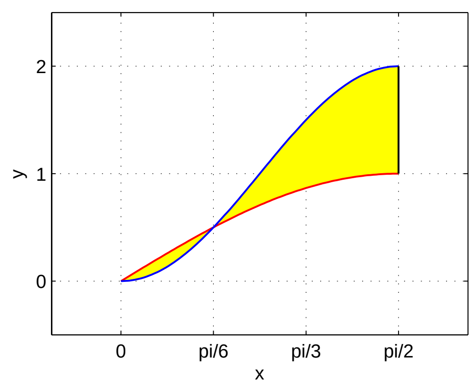
```
%-------------------------------------------------
% Stewart 470/44
%
syms x
f = (cos(x) + sin(x)) / sin(2*x); pretty(f)

                               cos(x) + sin(x)
                               ---------------
                                  sin(2 x)
F = int(f, x);
pretty(F)

   1/2 log(csc(x) - cot(x)) + 1/2 log(sec(x) + tan(x))
%
echo off; diary off
```

s470x50

Find the area of the region bounded by $y = \sin x$ and $y = 2\sin^2 x$, $x = 0$, and $x = \frac{\pi}{2}$.

Solution

- Here is a plot of the region.



- When the curves intersect we have

$$\begin{aligned} \sin x &= 2\sin^2 x \\ \sin x\,(1 - 2\sin x) &= 0 \\ \sin x &= 0, \tfrac{1}{2} \\ x &= 0, \tfrac{\pi}{6}, \end{aligned}$$

  recalling that $0 \le x \le \frac{\pi}{2}$.

- The area is

$$\begin{aligned} A &= \int_0^{\pi/6} \sin x - 2\sin^2 x \, dx + \int_{\pi/6}^{\pi/2} 2\sin^2 x - \sin x \, dx \\ &= 1 + \frac{\pi}{6} - \frac{\sqrt{3}}{2} \approx 0.66 \text{ cm}^2. \end{aligned}$$

- Here is the code that performs the computations and renders the graphics.

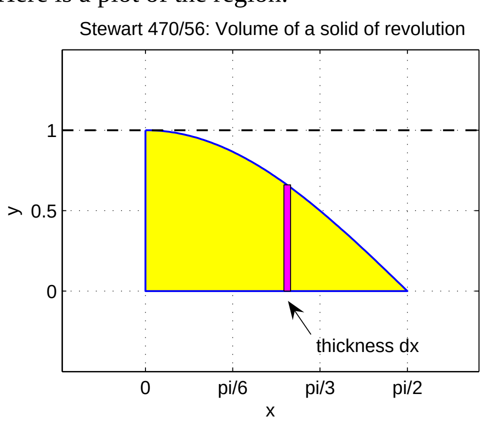
```
%-------------------------------------------------
% Stewart 470/50: Area between curves
%
syms x
f = sin(x); g = 2.* sin(x).^2;
intersections = solve(sin(x) - 2*sin(x)^2, x)
intersections =
      0
 1/6*pi
%
x = linspace(0, pi/2);
y1 = eval(f); y2 = eval(vectorize(g));
X = [x fliplr(x)];
Y = [y1 fliplr(y2)];
%
fill(X, Y, 'y', 'LineStyle', 'none')
grid on; hold on;
plot(x, y1, 'r', 'LineWidth', 1)
plot(x, y2, 'b', 'LineWidth', 1)
plot([pi/2 pi/2], [1 2], 'k', 'LineWidth', 1)
%
axis([-pi/8 5*pi/8, -0.5 2.5])
xlabel('x'); ylabel('y')
title('Stewart 470/50: Area between curves')
set(gca, 'Xtick', 0 : pi/6 : pi/2)
set(gca, 'XtickLabel', {'0';'pi/6';'pi/3';'pi/2'})
set(gca, 'Ytick', 0:2)
%
syms x
A = int(f-g, x, 0, pi/6)  +  int(g-f, x, pi/6, pi/2);
pretty(A)

                                       1/2
                           - 1/2 3    + 1/6 pi + 1

Area = eval(A)
Area =
    0.6576
%
echo off; diary off
```

s470x56

Find the volume of the region obtained by rotating the region bounded by $y = \cos x$, $y = 0$, $x = 0$, and $x = \frac{\pi}{2}$, about the line $y = 1$.

Solution

- Here is a plot of the region.



- The volume is

$$
\begin{aligned}
V &= \int_0^{\pi/2} \pi\,(1)^2 - \pi\,(1-y)^2\,dx \\
&= \int_0^{\pi/2} \pi\,(1)^2 - \pi\,(1-\cos x)^2\,dx \\
&= 2\pi - \frac{\pi^2}{4} \approx 3.82\text{ cm}^3.
\end{aligned}
$$

- Here is the code that performs the computations and renders the graphics.

```
%-------------------------------------------------
% Stewart 470/56: Volume of a solid of revolution
%
x = linspace(0, pi/2); y = cos(x);
X = [x 0 0]; Y = [y 0 1];
%
fill(X, Y, 'y', 'LineStyle', 'none')
grid on; hold on;
plot(X, Y, 'b', 'LineWidth', 1)
plot([-0.5 2], [1 1], 'k--', 'LineWidth', 1)
%
axis([-0.5 2 -0.5 1.5])
xlabel('x'); ylabel('y')
title('Stewart 470/56: Volume of a solid of revolution')
set(gca, 'Xtick', 0 : pi/6 : pi/2)
set(gca, 'XtickLabel', {'0';'pi/6';'pi/3';'pi/2'})
set(gca, 'Ytick', 0:0.5:1)
%
% Slice
xs = 0.85; ys = cos(xs); h = 0.02;
fill([xs-h xs+h xs+h xs-h xs-h], ...
    [0 0 ys ys 0], 'm')
%
syms x
V = int(pi*1^2  -  pi*(1 - cos(x))^2, x, 0, pi/2)
V =
-1/4*pi^2+2*pi
pretty(V)

                                2
                       - 1/4 pi   + 2 pi

Volume = eval(V)
Volume =
    3.8158
%
echo off; diary off
```