Spring 2005 Math 152
8 Techniques of Integration
8.9 Improper Integrals
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Summary

Preliminaries

Infinite intervals are those that have one of these forms.

$$\begin{aligned}(-\infty, b] &: \{x : x \le b\}\\ [a, \infty) &: \{x : x \ge a\}\\ (-\infty, \infty) &: \mathbb{R} = \text{all real numbers}\end{aligned}$$

The function f has an **infinite discontinuity** at $x = a$ if at least one of the following occurs. Basically, the function values of f become unbounded near $x = a$.

$$\lim_{x\to a^-} f(x) = \infty \qquad \lim_{x\to a^+} f(x) = \infty$$
$$\lim_{x\to a^-} f(x) = -\infty \qquad \lim_{x\to a^+} f(x) = -\infty$$

Improper integrals

These are definite integrals that involve infinite intervals of integration and/or integrands with infinite discontinuities. In each case below, if the limit in question exists, then the improper integral is said to **converge** or be **convergent**. Otherwise, it is said to **diverge** or be **divergent**.

Improper integrals involving infinite intervals

$$\begin{aligned}\int_a^\infty f(x)\,dx &= \lim_{t\to\infty}\int_a^t f(x)\,dx \quad (t \ge a)\\ \int_{-\infty}^b f(x)\,dx &= \lim_{t\to-\infty}\int_t^b f(x)\,dx \quad (t \le b)\\ \int_{-\infty}^\infty f(x)\,dx &= \int_{-\infty}^r f(x)\,dx + \int_r^\infty f(x)\,dx\end{aligned}$$

In the third case, both integrals on the right-hand-side of the equation must converge; r is a fixed real number.

Improper integrals involving an infinite discontinuity at $x = c$

$$\begin{aligned}\int_a^c f(x)\,dx &= \lim_{t\to c^-}\int_a^t f(x)\,dx \quad (a \le t < c)\\ \int_c^b f(x)\,dx &= \lim_{t\to c^+}\int_t^b f(x)\,dx \quad (c < t \le b)\\ \int_a^b f(x)\,dx &= \int_a^c f(x)\,dx + \int_c^b f(x)\,dx\end{aligned}$$

In the third case, both integrals on the right-hand-side of the equation must converge; here $a < c < b$.

Other improper integrals These involve *both* infinite intervals of integration *and* integrands with infinite discontinuities.

Comparison Theorem

Let f and g be continuous functions with $f \ge g \ge 0$ on (a, ∞).

(a) If $\int_a^\infty f(x)\,dx$ is convergent, then $\int_a^\infty g(x)\,dx$ is convergent. Moreover, $\int_a^\infty g(x)\,dx \le \int_a^\infty f(x)\,dx = L$.

(b) If $\int_a^\infty g(x)\,dx$ is divergent, then $\int_a^\infty f(x)\,dx$ is divergent. Furthermore, $\int_a^\infty g(x)\,dx = \int_a^\infty f(x)\,dx = \infty$.

(Just think of areas under curves and these assertions are clear!)

Note The integral $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ converges for $p > 1$ and diverges for $p \le 1$.

Numerical evaluation of improper integrals

Suppose that we have an improper integral involving an infinite interval of integration for which the integrand $f(x)$ has no closed-form antiderivative. To numerically approximate this integral, use the change of variables $w = \tan^{-1} x$ to transform the infinite interval of integration into a finite one. The inverse transformation is $x = \tan w$, whence $dx = \sec^2 w\,dw$. For example,

$$\int_1^\infty \frac{1}{\sqrt{x^3+1}}\,dx = \int_{\pi/4}^{\pi/2} \frac{\sec^2 w\,dw}{\sqrt{1+\tan^3 w}} \approx 1.8948.$$

See MATLAB Example **s517x52** below for details.

Hand Examples

516/4

Evaluate $\displaystyle\int_2^\infty \frac{1}{(x+3)^{3/2}}\,dx$.

Solution

As $t \to \infty$, we have

$$\int_2^t (x+3)^{-3/2}\,dx = \left(-2\,(x+3)^{-1/2}\right)\Big|_2^t = \frac{2}{\sqrt5} - \frac{2}{\sqrt{t+3}} \to \frac{2}{\sqrt5}.$$

So $\displaystyle\int_2^\infty \frac{1}{(x+3)^{3/2}}\,dx = \frac{2}{\sqrt5}$. The improper integral *converges*.

Example A

Evaluate $\int_{-\infty}^{\infty} x^3\,dx$.

Solution

As $t \to \infty$, we have $\int_0^t x^3\,dx = \frac{1}{4}x^4\Big|_0^t = \frac{1}{4}t^4 \to \infty$. Therefore, $\int_0^\infty x^3\,dx$ diverges. Thus $\int_{-\infty}^{\infty} x^3\,dx = \int_{-\infty}^{0} x^3\,dx + \int_0^\infty x^3\,dx$ *diverges* as well.

517/12alt

Evaluate $\int_{-\infty}^{\infty} x^3 e^{-x^4}\,dx$.

Solution

- Now $\int_0^t x^3 e^{-x^4}\,dx = \left(-\frac{1}{4}e^{-x^4}\right)\Big|_0^t = \frac{1}{4} - \frac{1}{4}e^{-t^4} \to \frac{1}{4}$ as $t \to \infty$.
- Let $w = -x$. Then $dw = -dx$ or $-dw = dx$ and thus

$$\begin{aligned}\int_{-\infty}^{0} x^3 e^{-x^4}\,dx &= -\int_{\infty}^{0} (-w)^3\, e^{-(-w)^4}\,dw \\ &= -\int_0^\infty w^3 e^{-w^4}\,dw = -\frac{1}{4}\end{aligned}$$

  in light of the preceding bullet.
- Therefore,

$$\begin{aligned}\int_{-\infty}^{\infty} x^3 e^{-x^4}\,dx &= \int_{-\infty}^{0} x^3 e^{-x^4}\,dx + \int_0^\infty x^3 e^{-x^4}\,dx \\ &= -\frac{1}{4} + \frac{1}{4} = 0.\end{aligned}$$

  The improper integral *converges* to 0.

517/20

Evaluate $\int_0^\infty x e^{-x}\,dx$.

Solution

- Let $u = x$ and $dv = e^{-x}dx$. Then $du = dx$ and $v = -e^{-x}$. So $\int x e^{-x}\,dx = -xe^{-x} + \int e^{-x}\,dx = -(x+1)\,e^{-x} + C$.
- Thus $\int_0^t x e^{-x}\,dx = 1 - (t+1)\,e^{-t} \to 1$ as $t \to \infty$ (see next bullet). Hence $\int_0^\infty x e^{-x}\,dx = 1$. The improper integral *converges* to 1.
- (Note that $\lim_{t\to\infty} \frac{t+1}{e^t} \overset{\text{L'H}}{=} \lim_{t\to\infty} \frac{1}{e^t} = 0$.)

517/29

Evaluate $\displaystyle\int_{-1}^{0} \frac{1}{x^2}\,dx$.

Solution

We have $\displaystyle\int_{-1}^{t} x^{-2}\,dx = \left(-x^{-1}\right)\Big|_{-1}^{t} = -\frac{1}{t} - 1 \to \infty$ as $t \to 0^-$. Therefore, $\displaystyle\int_{-1}^{0} \frac{1}{x^2}\,dx$ *diverges* to ∞.

517/30

Evaluate $\displaystyle\int_1^9 \frac{1}{\sqrt[3]{x-9}}\,dx$.

Solution

The integral *converges*.

$$\begin{aligned}\int_1^9 \frac{1}{\sqrt[3]{x-9}}\,dx &= \lim_{t\to 9^-} \int_1^t (x-9)^{-1/3}\,dx \\ &= \lim_{t\to 9^-} \frac{3}{2}(x-9)^{2/3}\Big|_1^t \\ &= \lim_{t\to 9^-} \left(\frac{3}{2}(t-9)^{2/3} - \frac{3}{2}(4)\right) = -6.\end{aligned}$$

517/50

Use the Comparison Theorem to determine whether the improper integral $\displaystyle\int_1^\infty \frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}}\,dx$ is convergent or divergent.

Solution

- Now $f(x) = \dfrac{\sqrt{1+\sqrt{x}}}{\sqrt{x}} \geq \dfrac{1}{\sqrt{x}} = x^{-1/2} = g(x) \geq 0$ for $x \geq 1$.
- As $t \to \infty$, we have

$$\int_1^t x^{-1/2}\,dx = 2x^{1/2}\Big|_1^t = 2\sqrt{t} - 2 \to \infty.$$

  Hence $\int_1^\infty x^{-1/2}\,dx$ diverges to ∞.
- By the Comparison Theorem, we conclude that $\displaystyle\int_1^\infty \frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}}\,dx$ is *divergent*.

517/56

Evaluate $\displaystyle\int_2^{\infty} \frac{1}{x\sqrt{x^2-4}}\,dx$.

Solution

The stated integral is improper for two reasons. First, the interval of integration is infinite. Second, as $x \to 2^+$,the integrand approaches $+\infty$. *Nonetheless, the integral appears to be remarkably easy to evaluate using a trig sub!*

- Let $x = 2\sec\theta$. Then $dx = 2\sec\theta\tan\theta\,d\theta$ and $\displaystyle\int_2^{\infty} \frac{1}{x\sqrt{x^2-4}}\,dx = \int_0^{\pi/2} \frac{2\sec\theta\tan\theta\,d\theta}{2\sec\theta\,(2\tan\theta)} = \int_0^{\pi/2} \frac{1}{2}\,d\theta = \frac{\pi}{4}.$
  This is the essence of the problem. But we are glossing over subtleties. . .
- More formally, let $f(\theta) = \dfrac{2\sec\theta\tan\theta}{2\sec\theta\cdot 2\tan\theta}$ for $0 < \theta < \frac{\pi}{2}$. Note that $f(\theta)$ is *not* defined for $\theta = 0, \frac{\pi}{2}$. On $\left(0, \frac{\pi}{2}\right)$, however, $f(\theta) = \frac{1}{2}$. Accordingly, the middle integral in the preceding bullet is *really* equivalent to

$$\begin{aligned}
&\lim_{\alpha\to 0^+}\int_{\alpha}^{\pi/4} f(\theta)\,d\theta + \lim_{\beta\to\frac{\pi}{2}^-}\int_{\pi/4}^{\beta} f(\theta)\,d\theta \\
=\ &\lim_{\alpha\to 0^+}\int_{\alpha}^{\pi/4} \frac{1}{2}\,d\theta + \lim_{\beta\to\frac{\pi}{2}^-}\int_{\pi/4}^{\beta} \frac{1}{2}\,d\theta \\
=\ &\lim_{\alpha\to 0^+}\left(\tfrac{1}{2}\left(\tfrac{\pi}{4}-\alpha\right)\right) + \lim_{\beta\to\frac{\pi}{2}^-}\left(\tfrac{1}{2}\left(\beta-\tfrac{\pi}{4}\right)\right) \\
=\ &\tfrac{1}{2}\left(\tfrac{\pi}{4}\right) + \tfrac{1}{2}\left(\tfrac{\pi}{4}\right) = \tfrac{\pi}{4}.
\end{aligned}$$

520/75

Use the Comparison Theorem to determine whether the improper integral $\displaystyle\int_1^{\infty} \frac{x^3}{x^5+2}\,dx$ is convergent or divergent.

Solution

- Now $f(x) = x^{-2} = \dfrac{x^3}{x^5} \geq \dfrac{x^3}{x^5+2} = g(x) \geq 0$ for $x \geq 1$.
- As $t \to \infty$, we have

$$\int_1^t x^{-2}\,dx = \left(-x^{-1}\right)\Big|_1^t = 1 - \frac{1}{t} \to 1.$$

  Hence $\int_1^{\infty} x^{-2}\,dx$ converges to 1.
- By the Comparison Theorem, we conclude that $\displaystyle\int_1^{\infty} \frac{x^3}{x^5+2}\,dx$ is *convergent.*

MATLAB Examples

MATLAB's **int** command can automatically handle improper integrals. For ∞, write **inf**; for $-\infty$, write **-int**.

s516x04

Evaluate $\displaystyle\int_2^{\infty} \frac{1}{(x+3)^{3/2}}\,dx$.

Solution

```
%
% Stewart 516/4
%
syms x
f = 1 / (x+3)^(3/2); pretty(f)

                                  1
                              ----------
                                     3/2
                              (x + 3)
I = int(f, x, 2, inf); pretty(I)

                                     1/2
                               2/5 5
I_float = eval(I)
I_float =
    0.8944
%
echo off; diary off
```

s89exA [Section 8.9, Example A, revisited]

Evaluate $\int_{-\infty}^{\infty} x^3\,dx$.

Solution

The result, **NaN** (the IEEE representation for Not-a-Number), signifies that the integral is divergent.

```
%
% Stewart 8.9/Ex A
%
syms x
f = x^3; pretty(f)

                                        3
                                       x
I = int(f, x, -inf, inf)
I =
NaN
%
echo off; diary off
```

s517x12alt

Evaluate $\int_{-\infty}^{\infty} x^3 e^{-x^4}\,dx$.

Solution

```
%
% Stewart 517/12alt
%
syms x
f = x^3 * exp(-x^4); pretty(f)

                                                   3        4
                                                  x  exp(-x )
I = int(f, x, -inf, inf); pretty(I)

                                                       0
%
echo off; diary off
```

s517x20

Evaluate $\int_0^\infty xe^{-x}\,dx$.

Solution

```
%
% Stewart 517/20
%
syms x
f = x * exp(-x); pretty(f)

                                                   x exp(-x)
I = int(f, x, 0, inf); pretty(I)

                                                       1
%
echo off; diary off
```

s517x29

Evaluate $\int_{-1}^{0} \frac{1}{x^2}\,dx$.

Solution

```
%
% Stewart 517/29
%
syms x
f = 1 / x^2; pretty(f)

                                                      1
                                                     ----
                                                       2
                                                      x
I = int(f, x, -1, 0); pretty(I)

                                                      Inf
%
echo off; diary off
```

s517x30

Evaluate $\int_1^9 \frac{1}{\sqrt[3]{x-9}}\,dx$.

Solution

If we attempt to use **int** to obtain an analytical solution, a complex result is returned. This is because MATLAB returns complex results for odd roots of negative numbers.

Accordingly, I wrote a function **root** that returns the real odd root of a negative number. We then numerically integrate the integral using MATLAB's **quad** routine, which employs an adaptive Simpson's method (as mentioned in Section 8.8).

```
%
% Stewart 517/30
%
I = quad(@f, 1, 9)
Warning: Divide by zero.
I =
   -6.0000
%
echo off; diary off
%--------------------
function y = f(x)
y = 1 ./ root(3, x-9);
%--------------------
function y = root(r,x)
y = sign(x) .* abs(x).^(1/r);
```

s517x52

Evaluate $\int_1^\infty \frac{1}{\sqrt{x^3+1}}\,dx$ *numerically*.

Solution

- In the corresponding hand example, we used the Comparison Theorem to show that the integral was convergent. Since the integrand has no closed-form antiderivative, we'd like to approximate the value of the integral numerically.
- Accordingly, let $w = \tan^{-1} x$. Then $x = \tan w$ and $dx = \sec^2 w\,dw$. This transforms the infinite interval of integration into a finite one. Hence

$$\int_1^\infty \frac{1}{\sqrt{x^3+1}}\,dx = \int_{\pi/4}^{\pi/2} \frac{\sec^2 w\,dw}{\sqrt{1+\tan^3 w}} \approx 1.8948.$$

```
%
% Stewart 520/52
%
I = quad(@g, pi/4, pi/2)
I =
    1.8948
%
echo off; diary off
%--------------------
function y = g(w)
y = sec(w).^2 ./ sqrt(1 + tan(w).^3);
```

s517x56

Evaluate $\int_2^\infty \frac{1}{x\sqrt{x^2-4}}\,dx$.

Solution

```
%
% Stewart 517/56
%
syms x
f = 1 / (x * sqrt(x^2 -4)); pretty(f)

                                                   1
                                            -------------
                                                2     1/2
                                            x (x  - 4)
I = int(f, x, 2, inf); pretty(I)

                                                1/4 pi

I_float = eval(I)
I_float =
    0.7854
%
echo off; diary off
```

s518x63

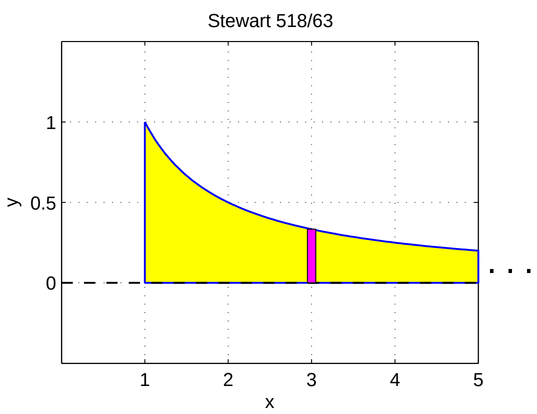
Consider the region $R = \{(x, y) : x \geq 1, 0 \leq y \leq 1/x\}$.

(a) Show that the area of R is infinite.

(b) However, if we rotate R about the x-axis, the volume of the resulting solid is *finite!*

Solution

Here is a figure that shows the region R. The MATLAB diary file follows that shows that the area of R is infinite, but that the volume of the solid of revolution is π: strange, but true!




```
%
% Stewart 518/63
%
syms x
A = int(1/x, x, 1, inf); pretty(A)

                                                     Inf

V = int(pi/x^2, x, 1, inf); pretty(V)

                                                     pi
%
echo off; diary off
```

s518x74

Find the value of the constant K for which the integral

$$\int_0^\infty \frac{x}{x^2+1} - \frac{K}{3x+1}\,dx$$

converges. Evaluate the integral for this value of K.

Solution

You really owe it to yourself to try this problem by hand to fully understand the subleties. That said, firepower has its appeal! We conclude that

$$\int_0^\infty \frac{x}{x^2+1} - \frac{K}{3x+1}\,dx = \begin{cases} \infty, & K < 3 \\ -\ln 3, & K = 3 \\ -\infty, & K > 3 \end{cases}$$

The **signum** or **sign** function is defined by

$$\operatorname{signum}(w) = \begin{cases} -1, & w < 0 \\ 0, & w = 0 \\ 1, & w > 0 \end{cases}.$$

```
%
% Stewart 518/74
%
syms K x
f = x/(x^2 + 1) - K/(3*x + 1); pretty(f)

                                  x         K
                               ------ - -------
                                 2      3 x + 1
                                x  + 1
I = int(f, x, 0, inf); pretty(I)

                              -signum(-3 + K) Inf
%
g = x/(x^2 + 1) - 3/(3*x + 1); pretty(g)

                                  x         3
                               ------ - -------
                                 2      3 x + 1
                                x  + 1
J = int(g, x, 0, inf); pretty(J)

                                     -log(3)
%
echo off; diary off
```

s520x75

Use the Comparison Theorem to determine whether the improper integral $\int_1^\infty \frac{x^3}{x^5+2}\,dx$ is convergent or divergent.

Solution

In the hand example, we showed that the integral was convergent via the Comparison Theorem. Indeed, MATLAB can analytically compute the value of the integral—but it's an inordinately complicated expression. Accordingly, we suppress it and merely show its floating point decimal approximation.

```
%
% Stewart 520/75
%
syms x
f = x^3 / (x^5 + 2); pretty(f)

                                           3
                                          x
                                        ------
                                         5
                                        x  + 2
I = int(f, x, 1, inf);
I_float = eval(I)
I_float =
    0.8268
%
echo off; diary off
```