

Spring 2005 Math 152

9 Further Applications of Integration

9.1 Differential Equations

Wed, 23/Feb ©2005, Art Belmonte

Summary

Terminology

- The **general form** of a **first-order differential equation** is

$$F(x, y, y') = 0.$$

Here x is the **independent variable**, y is the **dependent variable**, and y' is the first derivative of y with respect to x . (In applications, the independent variable will often be t for time.) The differential equation (DE) is assumed to hold on some open real interval I , possibly of infinite extent.

- If we are able to isolate y' in the general form, we obtain the **normal form** of the differential equation:

$$y' = f(x, y).$$

- An **explicit solution** is a function $y = \phi(x)$ which when substituted into the differential equation yields an identity for all x in I . This is shown via differentiation and direct substitution.
- An **implicit solution** is a relation $G(x, y) = 0$ which yields one or more explicit solutions on I , perhaps only in theory. This is usually shown via implicit differentiation.
- A **general solution** involves an arbitrary constant C or K .
- An **initial value problem (IVP)** consists of a differential equation in normal form together with an **initial condition (IC)**:

$$y' = \frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0.$$

Here x_0 and y_0 are constants. The solution to an initial value problem is called a **particular solution**.

Separable differential equation

This is a differential equation that has the form $\frac{dy}{dx} = \frac{g(x)}{h(y)}$ or

$\frac{dy}{dx} = g(x)f(y)$. Separate the expressions and differentials to obtain $h(y) dy = g(x) dx$. Then integrate (antidifferentiate) each side, $\int h(y) dy = \int g(x) dx$ to obtain $H(y) = G(x) + C$. This result implicitly defines y in terms of x ; i.e., we have an implicit solution. If we *can* subsequently solve for y (i.e., if H has an inverse function H^{-1}), then we can write the dependent variable y explicitly as a function of the independent variable x ; i.e., we have an explicit solution. *In this section all DEs are separable.*

Direction fields

One meaning of the normal form $y' = f(x, y)$ is “the slope of the tangent line at a point is given by this expression.” Do this for a bunch of lattice points in a rectangular region of the xy -plane and you obtain a **direction field** or **slope field**. This is hard to draw by hand, but quite easy to do using **dfield7** in MATLAB, a routine written by Professor John Polking at Rice University.

Mixing Problems

Mixing problems deal with the amount $x(t)$ of a substance in solution in a tank. The solution may be a salt solution, sugar solution, blood, etc. Our starting point is the classical balance law, which says that

$$\begin{aligned} \text{net rate} &= \text{rate in} - \text{rate out} \\ \frac{dx}{dt} &= \text{rate in} - \text{rate out} \end{aligned}$$

where the rates are given as flow rate \times concentration.

Logistic Growth

The **logistical differential equation** is a realistic way to model population growth.

$$\frac{dy}{dt} = ky(M - y)$$

Here M is the maximum size the population can reach (also known as the **carrying capacity**) and k is a proportionality constant. We'll use MATLAB to solve this differential equation given an initial population $y(0) = y_0$.

Hand Examples

533/4

Solve the differential equation $y' = xy$.

Solution

We have

$$\begin{aligned} \frac{dy}{dx} &= xy \\ \frac{1}{y} dy &= x dx \\ \ln |y| &= \frac{1}{2}x^2 + A \\ \pm y &= |y| = e^{A+x^2/2} = e^A e^{x^2/2} = B e^{x^2/2} \\ y &= C e^{x^2/2} \end{aligned}$$

To check your solution, explicitly differentiate and substitute back into the differential equation. You should obtain an identity in the independent variable. Equivalently, show that LHS–RHS = 0. Here LHS and RHS stand for the left- and right-hand sides of the original DE. (This is how to do it in MATLAB.)

$$y' - xy = Cxe^{x^2/2} - x \left(Ce^{x^2/2} \right) = 0$$

533/6

Solve the differential equation $y' = \frac{\ln x}{xy + xy^3}$.

Solution

We have

$$\begin{aligned} \frac{dy}{dx} &= \frac{\ln x}{xy + xy^3} = \left(\frac{\ln x}{x} \right) \left(\frac{1}{y + y^3} \right) \\ (y + y^3) dy &= \frac{\ln x}{x} dx \\ \frac{1}{2}y^2 + \frac{1}{4}y^4 &= \frac{1}{2}(\ln x)^2 + C \end{aligned}$$

To check your solution, implicitly differentiate and solve for $y' = dy/dx$. You should get the original differential equation.

$$\begin{aligned} yy' + y^3y' &= \frac{\ln x}{x} \\ (y + y^3)y' &= \frac{\ln x}{x} \\ y' &= \frac{\ln x}{x(y + y^3)} = \frac{\ln x}{xy + xy^3} \end{aligned}$$

534/12

Solve the initial value problem

$$x dx + 2y\sqrt{x^2 + 1} dy = 0, \quad y(0) = 1.$$

Solution

First obtain a general solution, then resolve the constant.

$$\begin{aligned} 2y dy &= -\left(x^2 + 1\right)^{-1/2} x dx \\ y^2 &= -\left(x^2 + 1\right)^{1/2} + C \\ 1 &= C - 1 \quad \text{[Substitute the IC.]} \\ C &= 2 \\ y^2 &= 2 - \sqrt{x^2 + 1} \quad \text{[implicit solution]} \\ y &= \sqrt{2 - \sqrt{x^2 + 1}} \quad \text{[explicit solution]} \end{aligned}$$

Notice that in the explicit solution we chose the *positive* square root so as to satisfy the initial condition $y(0) = 1$.

534/16

Find a function g such that $g'(x) = g(x)(1 + g(x))$ and $g(0) = 1$.

Solution

Let $y = g(x)$. Then $y' = dy/dx = y(y + 1)$. Hence

$$\begin{aligned} \frac{1}{y(y + 1)} dy &= 1 dx \\ \left(\frac{1}{y} - \frac{1}{y + 1} \right) dy &= 1 dx \quad \text{[via partial fractions, below]} \\ \ln y - \ln(y + 1) &= x + C \quad \text{[} y \text{ is initially near 1.]} \\ -\ln 2 &= C \quad \text{[Substitute the IC.]} \\ \ln\left(\frac{y}{y + 1}\right) &= x - \ln 2 \\ \frac{y}{y + 1} &= \frac{e^x}{2} \quad \text{[implicit solution]} \\ 2y &= e^x y + e^x \\ g(x) = y &= \frac{e^x}{2 - e^x} = \frac{1}{2e^{-x} - 1} \end{aligned}$$

Here are the details of the partial fraction decomposition.

$$\begin{aligned} \frac{1}{y(y + 1)} &= \frac{A}{y} + \frac{B}{y + 1} \\ 1 &= A(y + 1) + By \\ 0y + 1 &= (A + B)y + A \end{aligned}$$

Thus $A + B = 0$ and $A = 1$ whence $B = -A = -1$.

534/30+ [mixing problem; chemical engineering]

A tank contains 1000 L of pure water.

- Through a pipe, brine [a salt solution] that contains 0.05 kg of salt per liter of water enters the tank at a rate of 5 L/min.
- Through a second pipe, brine that contains 0.04 kg of salt per liter of water enters the tank at a rate of 10 L / min.
- The solution is kept well mixed and drains from the tank at a rate of 15 L/min.

- How much salt is in the tank after t minutes?
- How much salt is in the tank after 1 hour [60 minutes]?
- When is there 25 kg of salt in the tank?
- What is the limiting amount of salt in the tank?

Solution

- (a) Let $y = y(t)$ be the amount of salt in the tank at time t . Since the tank initially contains pure water, we have $y(0) = 0$ kg of salt in the tank at the start. The balance law from the Summary gives

$$\frac{dy}{dt} = \text{rate in} - \text{rate out}$$

$$\frac{dy}{dt} = \left(5 \frac{\text{L}}{\text{min}} \times \frac{1}{20} \frac{\text{kg}}{\text{L}}\right) + \left(10 \frac{\text{L}}{\text{min}} \times \frac{1}{25} \frac{\text{kg}}{\text{L}}\right) - \left(15 \frac{\text{L}}{\text{min}} \times \frac{y \text{ kg}}{1000 \text{ L}}\right)$$

$$\frac{dy}{dt} = \left(\frac{13}{20} - \frac{3y}{200}\right) \frac{\text{kg}}{\text{min}}$$

$$\frac{1}{130 - 3y} dy = \frac{1}{200} dt$$

$$-\frac{1}{3} \ln(130 - 3y) = \frac{1}{200}t + A \quad [y \text{ is initially near } 0]$$

$$\ln(130 - 3y) = -\frac{3}{200}t + B$$

$$130 - 3y = e^{-3t/200} e^B = C e^{-3t/200}$$

$$-3y = -130 + C e^{-3t/200}$$

$$y = \frac{130}{3} + K e^{-3t/200}$$

$$0 = \frac{130}{3} + K \quad [\text{Substitute the IC.}]$$

$$K = -\frac{130}{3}$$

$$y = \frac{130}{3} \left(1 - e^{-3t/200}\right)$$

- (b) When $t = 60$ minutes, we have

$$y = \frac{130}{3} \left(1 - e^{-180/200}\right) \approx 25.72 \text{ kg.}$$

- (c) When $y = 25$ kg, we have

$$25 = \frac{130}{3} \left(1 - e^{-3t/200}\right)$$

$$\frac{75}{130} = 1 - e^{-3t/200}$$

$$e^{-3t/200} = 1 - \frac{75}{130} = \frac{55}{130} = \frac{11}{26}$$

$$t = -\frac{200}{3} \ln\left(\frac{11}{26}\right) \approx 57.35 \text{ min}$$

- (d) As $t \rightarrow \infty$, we see that the limiting amount of salt is

$$\lim_{t \rightarrow \infty} y(t) = \lim_{t \rightarrow \infty} \frac{130}{3} \left(1 - e^{-3t/200}\right) = \frac{130}{3} \approx 43.33 \text{ kg.}$$

Remark This is a *lot* of work to do by hand! That's why we have computer algebra systems (CAS). See the corresponding MATLAB example for an exposition that employs firepower!

MATLAB Examples

MATLAB's **dsolve** command can automatically solve differential equations, with or without initial conditions. Or you may use the **deSolve** command on the TI-89. Plot explicit solutions with MATLAB's **plot** command. Plot implicit solutions with its **ezplot** or **contour** commands.

s533x04

Solve the differential equation $y' = xy$.

Solution

For MATLAB's **dsolve** command, we write **Dy** for y' or dy/dx . This is ultra-concise. The differential equation is entered as the first argument, a string. The second argument is the independent variable, again entered as a string. In the answer, $C1$ is a constant.

```
%
% Stewart 533/4
%
y = dsolve('Dy = x*y', 'x'); pretty(y)
%
% C1 exp(1/2 x^2)
%
echo off; diary off
```

s534x12

Solve the initial value problem

$$x dx + 2y\sqrt{x^2 + 1} dy = 0, \quad y(0) = 1.$$

Solution

The initial condition (another string) follows the differential equation in the **dsolve** command. Also, we divide the original DE by the differential dx so as to have dy/dx and thus **Dy**.

```
%
% Stewart 534/12
%
y = dsolve('x + 2*y*sqrt(x^2 + 1)*Dy = 0', ...
'y(0)=1', 'x');
pretty(y)
%
% (-x^2 + 1)^{1/2} + 2)^{1/2}
%echo off; diary off
```

s534x16

Find a function g such that $g'(x) = g(x)(1 + g(x))$ and $g(0) = 1$.

Solution

```

%
% Stewart 534/16
%
g = dsolve('Dg = g*(1+g)', 'g(0)=1', 'x');
pretty(g)

          1
-----
-1 + 2 exp(-x)

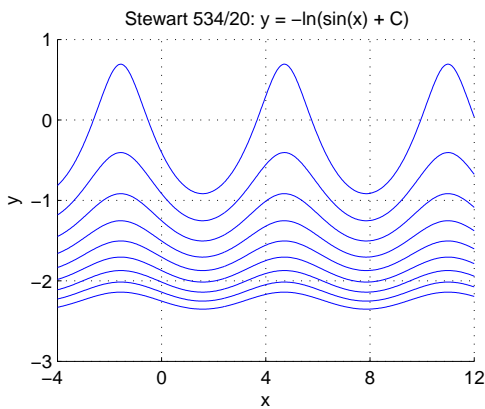
%
echo off; diary off

```

s534x20

Solve the differential equation $e^{-y}y' + \cos x = 0$ and graph several members of the family of solutions.

Solution



Here is a script M-file followed by a tiny diary file with the general solution, $y = -\ln(\sin x + C)$.

```

%
% Stewart 534/20
%
syms C1
y = dsolve('exp(-y)*Dy + cos(x) = 0', 'x');
pretty(y)

          -log(sin(x) + C1)

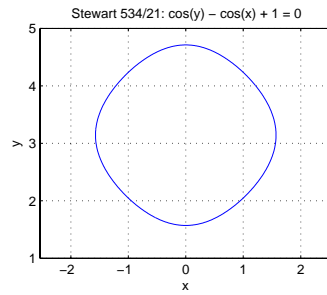
%
figure; grid on; hold on
x = linspace(-4, 12, 500);
for K = 1.5 : 1 : 9.5
    curve = subs(y, C1, K);
    Y = eval(vectorize(curve));
    plot(x,Y)
end
xlabel('x'); ylabel('y')
title('Stewart 534/20: y = -ln(sin(x) + C)')
axis([-4 12 -3 1])
set(gca, 'Xtick', -4 : 4 : 12)
set(gca, 'Ytick', -3:1)
%
echo off; diary off

```

s534x21

Solve the IVP $y' = \frac{\sin x}{\sin y}$, $y(0) = \frac{\pi}{2}$. Graph the solution.

Solution



```

%
% Stewart 534/21
%
y = dsolve('Dy = sin(x) / sin(y)', ...
    'y(0)=pi/2', 'x');
pretty(y) % explicit solution

          acos(cos(x) - 1)
% implicit solution; cos(y) = cos(x) - 1
% or equivalently cos(y) - cos(x) + 1 = 0
%
ezplot('cos(y) - cos(x) + 1', [-2 2 1 5])
grid on; axis equal
xlabel('x'); ylabel('y')
title('Stewart 534/21: cos(y) - cos(x) + 1 = 0')
set(gca, 'Xtick', -2:2)
set(gca, 'Ytick', 1:5)
%
echo off; diary off

```

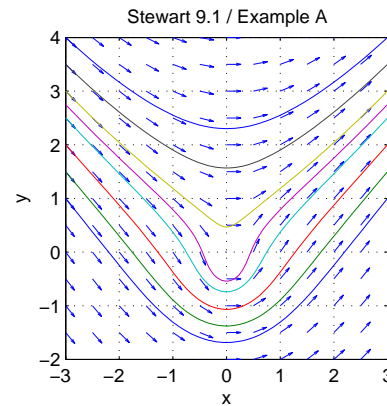
s91_exA

Sketch a direction field for the differential equation

$$y' = \frac{x}{\ln(1 + x^2 + y^2)}.$$

Then use it to sketch several solution curves.

Solution



Notice how the solution curves follow the “flow” of the direction field. There is a MATLAB routine called **dfield7**, written by Professor John Polking of Rice University, that let's you plot

direction fields and solution curves interactively using a GUI. It's quite intuitive; ask me about it if you are interested. The plot above was constructed more semiautomatically using **quiver** (for the direction field arrows) and **ode45** and **plot** (for the solution curves). The **ode45** routine solves differential equations numerically. You may subsequently plot the output.

```
%
% Stewart 9.1/Example A
%
x = -4 : 0.5 : 4; y = x;
[X Y] = meshgrid(x,y);
U = log(1 + X.^2 + Y.^2); V = X;
S = sqrt(U.^2 + V.^2);
W = U./S; Z = V./S;
Warning: Divide by zero.
Warning: Divide by zero.
quiver(X,Y,W,Z,0.4); hold on
%
xspan = [-3 3];
y0 = [1 1.5 2 2.5 2.75 3 3.5 4];
[x y] = ode45(@f, xspan, y0);
plot(x,y); grid on
axis equal; axis([-3 3 -2 4])
xlabel('x'); ylabel('y')
title('Stewart 9.1 / Example A')
%
echo off; diary off
```

s534x30p [534/30+ revisited: mixing problem; chemical engineering]

A tank contains 1000 L of pure water.

- Through a pipe, brine [a salt solution] that contains 0.05 kg of salt per liter of water enters the tank at a rate of 5 L/min.
- Through a second pipe, brine that contains 0.04 kg of salt per liter of water enters the tank at a rate of 10 L / min.
- The solution is kept well mixed and drains from the tank at a rate of 15 L/min.

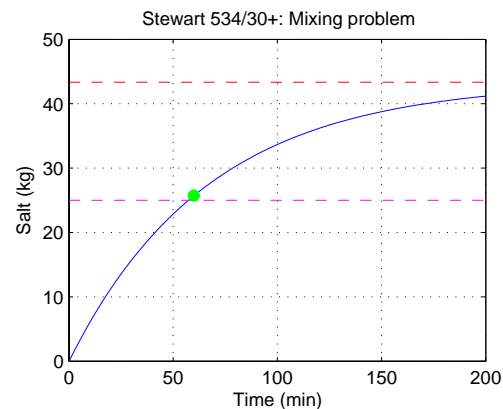
- How much salt is in the tank after t minutes?
- How much salt is in the tank after 1 hour [60 minutes]?
- When is there 25 kg of salt in the tank?
- What is the limiting amount of salt in the tank?

Solution

See the hand example for the formulation of the initial value problem. Once that is done, MATLAB rapidly dispatches the computations. An illustrative plot is also produced.

```
%
% Stewart 534/30
%
%
% (a)
syms t
```

```
y = dsolve('Dy = 13/20 - 3/200*y', 'y(0)=0', 't');
pretty(y)
130/3 - 130/3 exp(- 3/200 t)
% (b)
y_60 = subs(y, t, 60)
y_60 =
25.7153
% (c)
t_25 = solve(y-25, t); t_25 = eval(t_25)
t_25 =
57.3468
% (d)
y_inf = limit(y, t, inf), yi = eval(y_inf)
y_inf =
130/3
yi =
43.3333
% (e) Plot
t = linspace(0, 200);
y = eval(vectorize(y));
plot(t,y); grid on; hold on
% Events
plot([0 200], [yi yi], 'r--')
plot([0 200], [25 25], 'm--')
plot(60, y_60, 'go', 'MarkerFaceColor', 'g')
%
xlabel('Time (min)'); ylabel('Salt (kg)')
title('Stewart 534/30+: Mixing problem')
%
echo off; diary off
```



s535x36

Biologists stocked a lake with 400 fish and estimated the carrying capacity (the maximal population for the fish of that species in that lake) to be 10,000. The number of fish tripled in the first year.

- Assuming that the size of the fish population satisfies the logistic differential equation, find an expression for the size of the population after t years.
- How long will it take the population to increase to 5000?

Solution

- After t years, the fish population is $y(t) = \frac{10000}{1 + 24 \left(\frac{11}{36}\right)^t}$.
- The fish population reaches 5000 after 2.68 years.

A plot of the fish population vs time appears after this MATLAB diary file.

```

%
% Stewart 535/36
%
% (a)
syms k t
y = dsolve('Dy = k*y*(10000-y)', 'y(0)=400', 't');
pretty(y)

          10000
-----
      1 + 24 exp(-10000 k t)
eq = subs(y-1200, t, 1); pretty(eq)

          10000
-----
      1 + 24 exp(-10000 k)
k = solve(eq, k)
k =
-1/10000*log(11/36)
y = subs(y); pretty(y)

          10000
-----
          11
      1 + 24 exp(log(-- ) t)
          36

% (b)
t_5000 = solve(y-5000, t); t_5000 = eval(t_5000)
t_5000 =
    2.6805
% (c) Plot
t = linspace(0, 10);
y = eval(vectorize(y));
plot(t,y); grid on; hold on
axis([0 10 0 12000])
% Events
plot([0 10], [5000 5000], 'r--')
%
xlabel('Time (yr)'); ylabel('Fish population')
title('Stewart 535/36: Fish population in a lake')
%
echo off; diary off

```

