

# Spring 2005 Math 152

## 9 Further Applications of Integration

### 9.2 First-Order Linear Equations

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#### Summary

##### Terminology

A first-order **linear** differential equation has the form

$$a_1(x) \frac{dy}{dx} + a_0(x)y = b(x).$$

The **coefficients**  $a_0, a_1, b$  are constants or functions of the independent variable  $x$  alone; they do *not* depend on the dependent variable  $y$ . Moreover, note that both  $y$  and its derivative  $y'$  occur to the first power only. The equation is **homogeneous** if  $b(x) = 0$  and **nonhomogeneous** if  $b(x) \neq 0$ .

##### Conventional procedure (CP) to solve linear equations

1. You *MUST* put the DE in **standard linear form (SLF)**:

$$y' + P(x)y = Q(x)$$

2. Construct an **integrating factor**  $\mu(x) = \exp(\int P(x) dx)$ . Note that *any* antiderivative  $\int p(x) dx$  will do. So take the constant of integration to be zero.

3. Multiply the SLF by the integrating factor  $\mu$  to obtain  $\mu y' + \mu P y = \mu Q$ . Notice *by design* that  $(\mu y)' = \mu y' + \mu' y = \mu y' + \mu P y$  (via the Chain Rule). Putting these two together gives

$$(\mu(x)y(x))' = \mu(x)Q(x)$$

which is separable. That is, it's of the form  $dw/dx = g(x)$  and we may use Section 9.1 techniques!

4. Antidifferentiate (i.e., indefinitely integrate) to obtain  $\mu(x)y(x) = \int \mu(x)Q(x) dx + C$ . We therefore have

$$y(x) = \frac{1}{\mu(x)} \int \mu(x)Q(x) dx + \frac{C}{\mu(x)}.$$

This is an explicit **general solution** to the first-order linear DE.

This method works for both homogenous and nonhomogeneous first-order linear equations. In the summary above,  $y$  is the dependent variable; the independent variable (unspecified) may be taken to be  $x$ . In other problems,  $t$  (time) is the independent variable and  $x$  or  $y$  (or some other letter) the dependent variable. Just be aware of the *structure!*

##### Executive Summary of the CP!

1. SLF:  $y' + P y = Q$ .
2. IF:  $\mu = \exp(\int P)$ .
3.  $\mu y' + \mu P y = \mu Q$ , whence  $(\mu y)' = \mu Q$ .
4. Antidifferentiate; then isolate  $y$ .

#### Hand Examples

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Determine whether the differential equation  $x^2 y' - y + x = 0$  is linear.

##### Solution

Rewrite the equation as  $(x^2) y' + (-1) y = -x$ , which is linear.

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Determine whether the differential equation  $yy' = \sin x$  is linear.

##### Solution

The equation is *not* linear, since the coefficient of  $y'$  is  $y$ , an expression which contains the *dependent* variable.

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Solve the differential equation  $xy' + 2y = e^{x^2}, x > 0$ .

##### Solution

Let's go through the steps of the Conventional Procedure.

1. Put the DE into SLF:  $y' + \frac{2}{x} y = \frac{e^{x^2}}{x}$ . Here  $P(x) = \frac{2}{x}$ , the coefficient of  $y$  in the SLF.
2. Construct an integrating factor:  $\mu(x) = \exp(\int P(x) dx)$  or  $\exp(\int \frac{2}{x} dx) = e^{2 \ln x} = x^2$ .
3. Multiply the SLF by  $\mu$ :  $x^2 y' + 2xy = x e^{x^2}$  or  $(x^2 y)' = x e^{x^2}$ , as advertised!
4. Integrate to obtain  $x^2 y = \frac{1}{2} e^{x^2} + C$ . Therefore,  $y = \frac{\frac{1}{2} e^{x^2} + C}{x^2} = \frac{e^{x^2} + K}{2x^2}$  is a general solution.

##### Example A

Solve the IVP  $\sin x \frac{dy}{dx} + y \cos x = x \sin x, y(\frac{\pi}{2}) = 2$ .

##### Solution

What have we here? We're already at Step 3 of the CP!

$$\begin{aligned} (y \sin x)' &= x \sin x \\ y \sin x &= \sin x - x \cos x + C \\ y &= 1 - x \frac{\cos x}{\sin x} + \frac{C}{\sin x} \\ 2 &= 1 + C \quad [\text{using the IC}] \\ C &= 1 \end{aligned}$$

Therefore,  $y = 1 - x \cot x + \csc x$ .

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Solve the IVP  $xy' - 3y = x^2$ ,  $y(1) = 0$ .

**Solution**

- Put the DE into SLF:  $y' - \frac{3}{x}y = x$ . Here  $P(x) = -\frac{3}{x}$ , the coefficient of  $y$  in the SLF.
- Construct an integrating factor:  $\mu(x) = \exp(\int P(x) dx)$  or  $\exp(\int -\frac{3}{x} dx) = e^{-3 \ln x} = x^{-3}$ .
- Multiply the SLF by  $\mu$ :  $x^{-3}y' - 3x^{-4}y = x^{-2}$  or  $(x^{-3}y)' = x^{-2}$ .
- Integrate to obtain  $x^{-3}y = -x^{-1} + C$ . Therefore,  $y = Cx^3 - x^2$  is a general solution.
- Now use the IC to resolve the constant  $C$ .

$$\begin{aligned} 0 &= C - 1 \\ C &= 1 \\ y &= x^3 - x^2 \end{aligned}$$

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Solve the differential equation  $y' + (\cos x)y = \cos x$  and graph several members of the family of solutions.

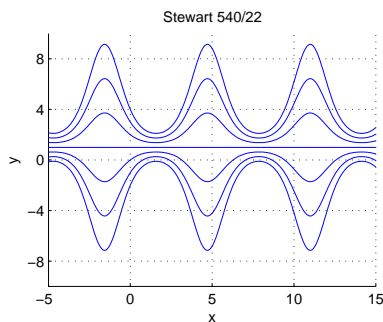
**Solution**

- The differential equation is already in standard linear form.
- Construct an integrating factor:  $\mu(x) = \exp(\int P(x) dx)$  or  $\exp(\int \cos dx) = e^{\sin x}$ .
- Multiply the SLF by  $\mu$ .

$$\begin{aligned} e^{\sin x}y' + (e^{\sin x} \cos x)y &= e^{\sin x} \cos x, \text{ or} \\ (e^{\sin x}y)' &= e^{\sin x} \cos x \end{aligned}$$

- Integrate to obtain  $e^{\sin x}y = e^{\sin x} + C$ . Therefore,  $y = 1 + Ce^{-\sin x}$  is a general solution.

Some solution curves are plotted below. See MATLAB for details.



**MATLAB Examples**

MATLAB's **dsolve** command can automatically solve differential equations, with or without initial conditions. Or you may use the **deSolve** command on the TI-89. Plot explicit solutions with MATLAB's **plot** command. Plot implicit solutions with its **ezplot** or **contour** commands.

**s540x08**

Solve the differential equation  $xy' + 2y = e^{x^2}$ ,  $x > 0$ .

**Solution**

For MATLAB's **dsolve** command, we write **Dy** for  $y'$  or  $dy/dx$ . This is ultra-concise. The differential equation is entered as the first argument, a string. The second argument is the independent variable, again entered as a string. In the answer  $C1$  is a constant.

```
%
% Stewart 540/8
%
syms x
y = dsolve('x*Dy + 2*y = exp(x^2)', 'x');
pretty(y)

1/2 exp(x^2) + C1
-----
x

%
echo off; diary off
```

**s92\_exA**

Solve the IVP  $\sin x \frac{dy}{dx} + y \cos x = x \sin x$ ,  $y(\frac{\pi}{2}) = 2$ .

**Solution**

```
%
% Stewart 9.2/Example A
%
syms x
y = dsolve('Dy*sin(x) + y*cos(x) = x*sin(x)', ...
'y(pi/2)=2', 'x');
pretty(y)

sin(x) - x cos(x) + 1
-----
sin(x)

%
echo off; diary off
```

**s540x16**

Solve the IVP  $xy' - 3y = x^2$ ,  $y(1) = 0$ .

**Solution**

```
%
% Stewart 540/16
%
syms x
y = dsolve('x*Dy - 3*y = x^2', 'y(1)=0', 'x');
y = simple(y); pretty(y)
```

