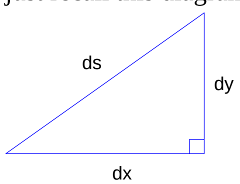
Spring 2005 Math 152
9 Further Applications of Integration
9.3 Arc Length
Mon, 28/Feb ©2005, Art Belmonte

Summary

Differential Triangle

Sir Isaac reminds us that to remember the formulas in this section, just recall this diagram.



The arc length differential $ds = \sqrt{dx^2 + dy^2}$ represents a differential length of arc along a curve in the xy-plane. Add them all up and you get the full length of the curve, $L = \int ds$. Let's look at how this works out for parametric and Cartesian curves. We play fast and loose with differentials.

Parametric curve (in terms of *t*)

For the parametric curve $\mathbf{r}(t) = [x(t), y(t)]$, $a \le t \le b$, we have

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{\left(\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right) dt^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

Thus $L = \displaystyle\int ds = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$

Cartesian curve (in terms of *x*)

For the Cartesian curve $y = f(x)$, $a \le x \le b$, we have

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{\left(\left(\frac{dx}{dx}\right)^2 + \left(\frac{dy}{dx}\right)^2\right) dx^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

Thus $L = \displaystyle\int ds = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$

Cartesian curve (in terms of y)

For the Cartesian curve $x = g(y)$, $c \le y \le d$, we have

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{\left(\left(\frac{dx}{dy}\right)^2 + \left(\frac{dy}{dy}\right)^2\right) dy^2} = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy.$$

Thus $L = \displaystyle\int ds = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy.$

Arc length functions

If we replace the *constant* upper limit of integration in any of the foregoing formulas with a *variable* one and change the dummy variable of integration, we obtain an **arc length function**.

For example, if $y = f(x)$ and the starting point is $P(a, f(a))$, then the arc length from P to $Q(x, f(x)$ is given by

$$s(x) = \int_a^x \sqrt{1 + (f'(w))^2}\, dw$$

This represents the [directed] arc length from a specified starting point on the curve to another arbitrary point. Also observe that

$$s(x) \text{ is } \begin{cases} \text{negative} & \text{for } x < a, \\ \text{zero} & \text{for } x = a, \\ \text{positive} & \text{for } x > a. \end{cases}$$

(See **s547x21** in the MATLAB examples for an illustration.)

Hand Examples

Let lengths be measured in centimeters (cm).

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Find the arc length of the curve

$$x = 3t - t^3, \quad y = 3t^2, \quad 0 \le t \le 2.$$

Solution

We use the arc length formula for a parametric curve.

$$\begin{aligned} L = \int ds &= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\ &= \int_0^2 \sqrt{(3 - 3t^2)^2 + (6t)^2}\, dt \\ &= \int_0^2 \sqrt{9 + 18t^2 + 9t^4}\, dt \\ &= 3\int_0^2 1 + t^2\, dt \\ &= 3\left(t + \tfrac{1}{3}t^3\right)\Big|_0^2 = 3\left(2 + \tfrac{8}{3}\right) - 0 = 14 \text{ cm.} \end{aligned}$$

546/8

Find the arc length of the curve

$$y = \frac{x^3}{6} + \frac{1}{2x}, \quad 1 \le x \le 2.$$

Solution

We use an arc length formula for a Cartesian curve.

$$
\begin{aligned}
L = \int ds &= \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \\
&= \int_1^2 \sqrt{1 + \left(\frac{x^2}{2} - \frac{x^{-2}}{2}\right)^2}\, dx \\
&= \int_1^2 \sqrt{\frac{x^4}{4} + \frac{2}{4} + \frac{x^{-4}}{4}}\, dx \\
&= \int_1^2 \frac{x^2 + x^{-2}}{2}\, dx \\
&= \tfrac{1}{2}\left(\tfrac{1}{3}x^3 - x^{-1}\right)\Big|_1^2 \\
&= \tfrac{1}{2}\left(\tfrac{8}{3} - \tfrac{1}{2}\right) - \tfrac{1}{2}\left(\tfrac{1}{3} - 1\right) \\
&= \tfrac{1}{2}\left(\tfrac{7}{3}\right) + \tfrac{1}{4} \\
&= \frac{14+3}{12} = \frac{17}{12} \approx 1.42 \text{ cm.}
\end{aligned}
$$

546/10

Find the arc length of the curve

$$y = \ln(\sin x), \quad \tfrac{\pi}{6} \le x \le \tfrac{\pi}{3}.$$

Solution

We use an arc length formula for a Cartesian curve.

$$
\begin{aligned}
L = \int ds &= \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \\
&= \int_{\pi/6}^{\pi/3} \sqrt{1 + \left(\frac{\cos x}{\sin x}\right)^2}\, dx \\
&= \int_{\pi/6}^{\pi/3} \csc x\, dx \\
&= \ln|\csc x - \cot x|\,\Big|_{\pi/6}^{\pi/3} \\
&= \ln\left(\frac{2}{\sqrt{3}} - \frac{1}{\sqrt{3}}\right) - \ln\left(2 - \sqrt{3}\right) \\
&= = \ln\left(\frac{1}{\sqrt{3}}\right) - \ln\left(2 - \sqrt{3}\right) \quad \text{[OK to stop here.]} \\
&= \ln\left(3^{-1/2}\right) - \ln\left(\frac{2-\sqrt{3}}{1} \cdot \frac{2+\sqrt{3}}{2+\sqrt{3}}\right) \\
&= \ln\left(2+\sqrt{3}\right) - \tfrac{1}{2}\ln 3 \approx 0.77 \text{ cm} \quad \text{[MATLAB]}
\end{aligned}
$$

Example A

Find the arc length of the curve $x = \frac{2}{3}(y-1)^{3/2}$, $1 \le y \le 5$.

Solution

We use an arc length formula for a Cartesian curve.

$$
\begin{aligned}
L = \int ds &= \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy \\
&= \int_1^5 \sqrt{1 + \left(\sqrt{y-1}\right)^2}\, dx \\
&= \int_1^5 y^{1/2}\, dy \\
&= \tfrac{2}{3} y^{3/2}\Big|_1^5 \\
&= \tfrac{2}{3}\left(5\sqrt{5}\right) - \tfrac{2}{3} \\
&= \frac{2}{3}\left(5\sqrt{5} - 1\right) \approx 6.79 \text{ cm.}
\end{aligned}
$$

MATLAB Examples

s546x12

Use Simpson's Rule with $n = 10$ to estimate the arc length of the curve $y = \tan x$, $0 \le x \le \frac{\pi}{4}$.

Solution

Use the **simp** routine I wrote to approximate the arc length.

```
%
% Stewart 546/12
%
syms x
y = tan(x);
hh = sqrt(1 + diff(y,x)^2); pretty(hh)

                                            2 2 1/2
                              (1 + (1 + tan(x) ) )
hh_equiv = sqrt(1 + sec(x)^4); pretty(hh_equiv)

                                          4 1/2
                                 (1 + sec(x) )
L = simp(@h, 0, pi/4, 10, 0)
L =
    1.2780
%
echo off; diary off
%--------------------
function z = h(x)
z = sqrt(1 + sec(x).^4);
```

s547x18

Find the exact arc length of the curve $x = \ln\left(1 - y^2\right)$, $0 \le y \le \frac{1}{2}$.

Solution

A CAS makes this easy. The arc length is $\ln 3 - \frac{1}{2} \approx 0.60$ cm.

```
%
% Stewart 547/18
%
syms y
x = log(1-y^2); pretty(x)

                                            2
                                   log(1 - y )
h = sqrt(1 + diff(x,y)^2); pretty(h)

                               /          2    \1/2
                               |         y     |
                               |1 + 4 ---------|
                               |             2 2|
                               \      (1 - y ) /
L = int(h, y, 0, 1/2); pretty(L)

                                 - 1/2 + log(3)

L_float = eval(L)
L_float =
    0.5986
%
echo off; diary off
```

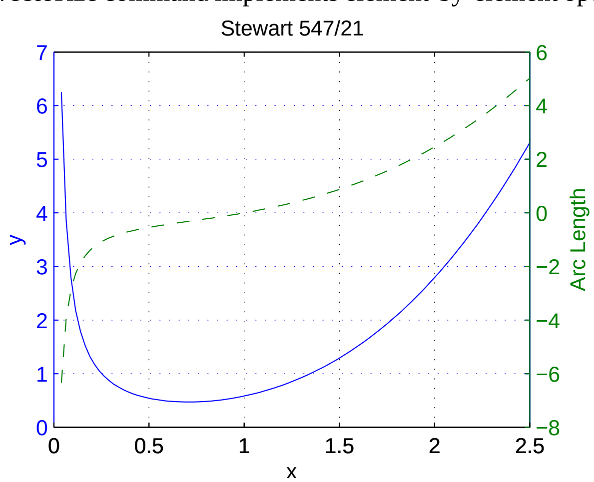
s547x21

- Graph the curve $y = \dfrac{x^3}{3} + \dfrac{1}{4x}, \; x > 0$.
- Find the arc length function for this curve with the starting point $P\left(1, \frac{7}{12}\right)$.
- Graph the arc length function.

Solution

The arc length function is $s(x) = \dfrac{x^3}{3} - \dfrac{1}{4x} - \dfrac{1}{12}, \; x > 0$.

MATLAB renders the needful. Note the fancy **plotyy** figure. The **vectorize** command implements element-by-element operations.



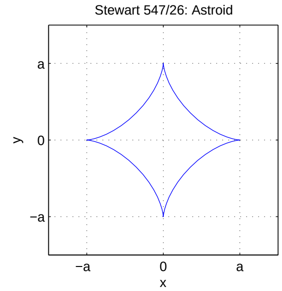

```
%
% Stewart 547/21
%
syms w x
y = w^3/3 + 1/(4*w); pretty(y)

                                    3
                               1/3 w  + 1/4 1/w
h = expand( simple( sqrt(1 + diff(y,w)^2) ) );
pretty(h)

                                      1      2
                                 1/4 ---- + w
                                       2
                                      w
ArcLength = expand(int(h, w, 1, x));
%
curve = subs(y, w, x); pretty(curve)

                                    3
                               1/3 x  + 1/4 1/x
pretty(ArcLength)

                                              3
                         - 1/4 1/x + 1/3 x  - 1/12
x = linspace(0.04, 2.5);
gc = eval(vectorize(curve));
gA = eval(vectorize(ArcLength));
[AX H1 H2] = plotyy(x,gc, x,gA);
grid on
xlabel('x'); title('Stewart 547/21')
set(get(AX(1), 'Ylabel'), 'String', 'y')
set(get(AX(2), 'Ylabel'), 'String', 'Arc Length')
set(H2, 'LineStyle', '--')
%
echo off; diary off
```

s547x26

Find the total length of the astroid $x = a\cos^3\theta$, $y = a\sin^3\theta$.

Solution

The astroid is traced out as θ varies from 0 to 2π.




```
%
% Stewart 547/26g
%
t = linspace(0, 2*pi, 73);
x = cos(t).^3; y = sin(t).^3;
plot(x,y); grid on; axis equal
axis([-1.5 1.5 -1.5 1.5])
xlabel('x'); ylabel('y')
title('Stewart 547/26: Astroid')
set(gca, 'Xtick', -1:1)
set(gca, 'Ytick', -1:1)
set(gca, 'XTickLabel', '-a|0|a')
set(gca, 'YTickLabel', '-a|0|a')
%
echo off; diary off
```

The total length of the astroid is $6a$.

```
%
% Stewart 547/26
%
syms a t positive
x = a*cos(t)^3; y = a*sin(t)^3;
h = simple( sqrt(diff(x,t)^2 + diff(y,t)^2) ); pretty(h)

                                 2                 2 1/2
                       3/4 (-2 a~   cos(4 t~) + 2 a~  )
L = simple(int(h, t, 0, 2*pi));
syms a t unreal; pretty(L)

                                      6 a
%
echo off; diary off
```

s547x29

- Graph the epitrochoid with equations

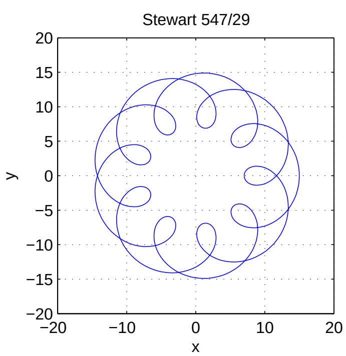
$$\begin{aligned} x &= 11\cos t - 4\cos(11t/2) \\ y &= 11\sin t - 4\sin(11t/2) \end{aligned}$$

  What parameter interval gives the complete curve?

- Use your CAS to find the approximate length of this curve.

Solution

The equations each have period 4π. So let $0 \leq \theta \leq 4\pi$. The arc length of the epitrochoid is approximately 294 cm.




```
%
% Stewart 547/29
%
syms t
x = 11*cos(t) - 4*cos(11*t/2);
y = 11*sin(t) - 4*sin(11*t/2);
hh = simple( sqrt(diff(x,t)^2 + diff(y,t)^2) ); pretty(hh)

                                                   1/2
                           11 (5 - 4 cos(9/2 t))
L = quad(@h, 0, 4*pi)
L =
  294.0277
%echo off; diary off
%--------------------
function z = h(t)
z = 11 * sqrt(5 - 4*cos(9*t/2));
```

Here we have used MATLAB's **quad** command to numerically compute the arc length integral. It uses an adaptive Simpson's Rule method. This involves variable step sizes as opposed to a fixed one.