Spring 2005 Math 152
9 Further Applications of Integration
9.5 Moments and Centers of Mass
Fri, 04/Mar ©2005, Art Belmonte

Summary

The point on which a thin plate balances horizontally is called the **center of mass** or **center of gravity** of the plate. Before we deal with this situation (where mass is distributed continuously), let's consider the simpler case of discrete masses.

Discrete masses

Let's consider a system of n discrete masses m_i located at points (x_i, y_i) in the xy-plane, $i = 1, \ldots, n$.

- **Total mass**: $m = \sum_{i=1}^{n} m_i$.
- **Moment about y-axis**: $M_y = \sum_{i=1}^{n} m_i x_i$.
- **Moment about x-axis**: $M_x = \sum_{i=1}^{n} m_i y_i$.
- **Center of mass, CM**: $(\bar{x}, \bar{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m} \right)$.

Continuously distributed mass

Now consider a flat plate (or **lamina**) with *uniform* density ρ that occupies a region $R = \{(x, y) : a \le x \le b, g(x) \le y \le f(x)\}$ in the xy-plane. (Here ρ is a *constant*.)

- **Total mass**: $m = \rho \int_a^b f(x) - g(x)\, dx$.
- **Moment about y-axis**: $M_y = \rho \int_a^b x \Big(f(x) - g(x) \Big)\, dx$.
- **Moment about x-axis**:

$$M_x = \frac{\rho}{2} \int_a^b \left(f^2(x) - g^2(x) \right) dx.$$

- **Center of mass, CM**: $(\bar{x}, \bar{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m} \right)$.

Theorem of Pappus

Let R be a plane region lying entirely on one side of a line L in a plane. Let A be the area of R, $P\ (\bar{x}, \bar{y})$ the center of mass of R, and r the distance from P to L. Then the volume V of the solid of revolution obtained by rotating R about L is $V = 2\pi r L$.

NOTE In this section, flat plates have *uniform* density $\rho = k$, a *constant*. In this situation, another word for center of mass is **centroid**. (It later courses you'll deal with *variable* density.)

Hand Examples

560/4

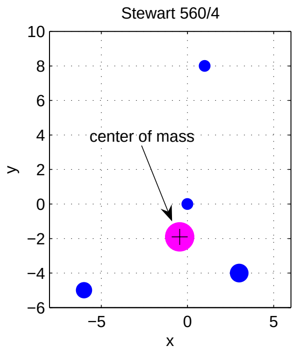
Given masses $m_1 = 3$, $m_2 = 3$, $m_3 = 8$, $m_4 = 6$, located at points

$$P_1\ (0, 0)\,, \quad P_2\ (1, 8)\,, \quad P_3\ (3, -4)\,, \quad P\ (-6, -5)\,,$$

find moments M_x and M_y along with center of mass CM $= (\bar{x}, \bar{y})$.

Solution

Here is a diagram showing the masses and the center of mass.



We compute the mass, moments, and center of mass. (Also see the corresponding MATLAB example.)

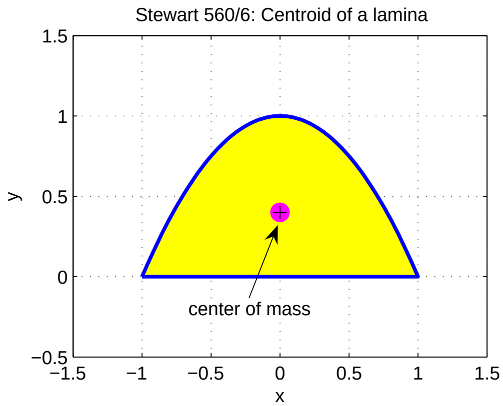
$$
\begin{aligned}
m &= \sum_{i=1}^{n} m_i = 3 + 3 + 8 + 6 = 20 \\
M_y &= \sum_{i=1}^{n} m_i x_i = \begin{bmatrix} 3 & 3 & 8 & 6 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 3 \\ -6 \end{bmatrix} = 0 + 3 + 24 - 36 = -9 \\
M_x &= \sum_{i=1}^{n} m_i y_i = \begin{bmatrix} 3 & 3 & 8 & 6 \end{bmatrix} \begin{bmatrix} 0 \\ 8 \\ -4 \\ -5 \end{bmatrix} = 0 + 24 - 32 - 30 = -38 \\
\text{CM} &= (\bar{x}, \bar{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m} \right) = \left(\frac{-9}{20}, \frac{-38}{20} \right) = (-0.45, -1.90)
\end{aligned}
$$

560/6

Find the centroid (center of mass of a flat plate of uniform density) of the region bounded by the curves $y = 1 - x^2$ and $y = 0$.

Solution

Here is a diagram showing the region and its center of mass.



We compute the mass, moments, and center of mass. (Also see the corresponding MATLAB example.)

$$
\begin{aligned}
m &= \rho \int_a^b f(x) - g(x)\, dx = \rho \int_{-1}^{1} 1 - x^2 - 0\, dx = \tfrac{4}{3}\rho \\
M_y &= \rho \int_a^b x\Big(f(x) - g(x)\Big)\, dx = \rho \int_{-1}^{1} x\left(1 - x^2\right)\, dx = 0 \\
M_x &= \frac{\rho}{2} \int_a^b \Big(f^2(x) - g^2(x)\Big)\, dx = \frac{\rho}{2} \int_{-1}^{1} \left(1 - x^2\right)^2 - 0^2\, dx = \tfrac{8}{15}\rho \\
\text{CM} &= (\bar{x}, \bar{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m}\right) = \left(\frac{0}{\frac{4}{3}\rho}, \frac{\frac{8}{15}\rho}{\frac{4}{3}\rho}\right) = \left(0, \frac{2}{5}\right) = (0.00, 0.40)
\end{aligned}
$$

Here are the integration details.

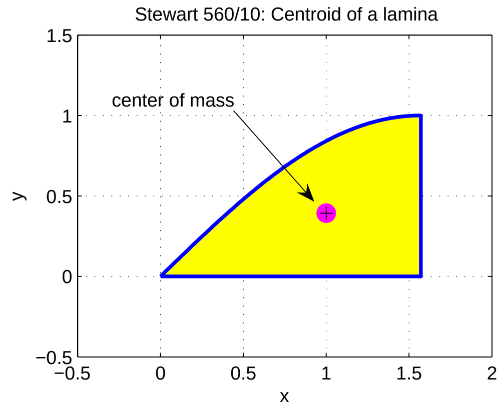
- $\int_{-1}^{1} 1 - x^2\, dx = 2\int_0^1 1 - x^2\, dx = 2\left(x - \frac{1}{3}x^3\right)\Big|_0^1 = \frac{4}{3}$
- $\int_{-1}^{1} x\left(1 - x^2\right)\, dx = 0$ since the integrand is odd and the interval of integration symmetric about 0.
- $\int_{-1}^{1} \left(1 - x^2\right)^2\, dx = 2\int_0^1 1 - 2x^2 + x^4\, dx$
  $= 2\left(x - \frac{2}{3}x^3 + \frac{1}{5}x^5\right)\Big|_0^1 = 2\left(\dfrac{15 - 10 + 3}{15}\right) = \dfrac{16}{15}$

560/10

Find the centroid (center of mass of a flat plate of uniform density) of the region bounded by the curves $y = \sin x$, $y = 0$, $x = 0$, and $x = \frac{\pi}{2}$.

Solution

Here is a diagram showing the region and its center of mass.



We compute the mass, moments, and center of mass. (Also see the corresponding MATLAB example.)

$$
\begin{aligned}
m &= \rho \int_a^b f(x) - g(x)\, dx = \rho \int_0^{\pi/2} \sin x - 0\, dx = \rho \\
M_y &= \rho \int_a^b x\Big(f(x) - g(x)\Big)\, dx = \rho \int_0^{\pi/2} x \sin x\, dx = \rho \\
M_x &= \frac{\rho}{2} \int_a^b \Big(f^2(x) - g^2(x)\Big)\, dx = \frac{\rho}{2} \int_0^{\pi/2} \sin^2 x - 0^2\, dx = \tfrac{1}{8}\pi\rho \\
\text{CM} &= (\bar{x}, \bar{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m}\right) = \left(\frac{\rho}{\rho}, \frac{\frac{1}{8}\pi\rho}{\rho}\right) = \left(1, \frac{\pi}{8}\right) = (1.00, 0.39)
\end{aligned}
$$

Here are the integration details.

- $\int_0^{\pi/2} \sin x\, dx = (-\cos x)\Big|_0^{\pi/2} = 0 - (-1) = 1$
- Let $u = x$ and $dv = \sin x\, dx$. Then $du = dx$ and $v = -\cos x$. Hence

  $$\int x \sin x\, dx = -x\cos x + \int \cos x\, dx = \sin x - x\cos x + C.$$

  Thus $\int_0^{\pi/2} x \sin x\, dx = (\sin x - x\cos x)\Big|_0^{\pi/2} = 1 - 0 = 1.$
- $\int_0^{\pi/2} \sin^2 x\, dx = \frac{1}{2}\int_0^{\pi/2} 1 - \cos 2x\, dx$
  $= \frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right)\Big|_0^{\pi/2} = \frac{\pi}{4}.$

MATLAB Examples

s560x04 [560/4 revisited]

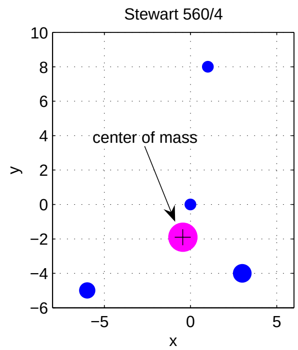
Given masses $m_1 = 3$, $m_2 = 3$, $m_3 = 8$, $m_4 = 6$, located at points

$$P_1\,(0, 0)\,, \quad P_2\,(1, 8)\,, \quad P_3\,(3, -4)\,, \quad P\,(-6, -5)\,,$$

find moments M_x and M_y along with center of mass $\text{CM} = (\bar{x}, \bar{y})$.

Solution

Here is a diagram showing the masses and the center of mass.



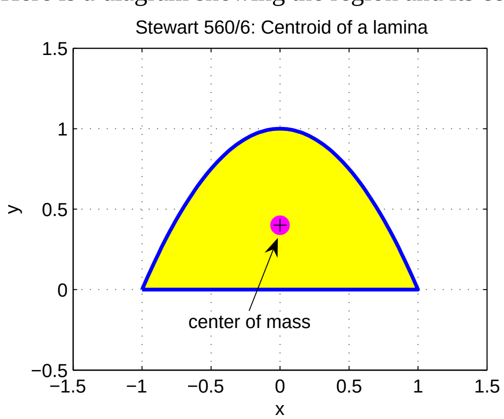

```
%
% Stewart 560/4
%
m = [3 3 8 6];                    % masses
P = [0 0; 1 8; 3 -4; -6 -5];      % points
x = P(:,1); y = P(:,2);           % x- and y-coordinates
mass = sum(m)                     % total mass
mass =
     20
My = m * x                        % matrix multiplication
My =
     -9
Mx = m * y                        % a.k.a., dot product
Mx =                              % [in this problem]
    -38
CM = [My Mx] / mass               % center of mass
CM =
    -0.4500   -1.9000
%
echo off; diary off
```

s560x06 [560/6 revisited]

Find the centroid (center of mass of a flat plate of uniform density) of the region bounded by the curves $y = 1 - x^2$ and $y = 0$.

Solution

Here is a diagram showing the region and its center of mass.



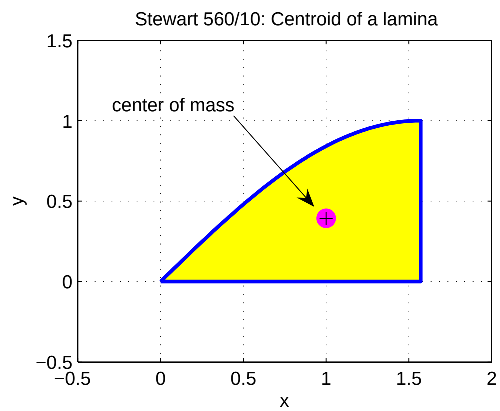

```
%
% Stewart 560/6
%
syms d x
m = d*int(1-x^2, x, -1, 1); pretty(m)

                                                     4/3 d
My = d*int(x*(1-x^2), x, -1, 1); pretty(My)

                                                        0
Mx = d/2*int((1-x^2)^2, x, -1, 1); pretty(Mx)

                                                    8/15 d
CM = [My Mx] / m; pretty(CM)

                                                 [0    2/5]
%
echo off; diary off
```

s560x10 [560/10 revisited]

Find the centroid (center of mass of a flat plate of uniform density) of the region bounded by the curves $y = \sin x$, $y = 0$, $x = 0$, and $x = \frac{\pi}{2}$.

Solution

Here is a diagram showing the region and its center of mass.



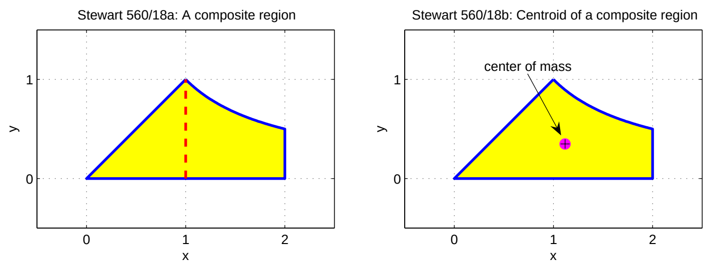

```
%
% Stewart 560/10
%
syms d x
m = d*int(sin(x), x, 0, pi/2); pretty(m)

                                                         d
My = d*int(x*sin(x), x, 0, pi/2); pretty(My)

                                                         d
Mx = d/2*int(sin(x)^2, x, 0, pi/2); pretty(Mx)

                                                   1/8 d pi
CM = [My Mx] / m; pretty(CM)

                                                 [1    1/8 pi]
CM = eval(CM)
CM =
    1.0000    0.3927
%
echo off; diary off
```

s560x18

Find the centroid of the region bounded by $y = x$, $y = 0$, $y = 1/x$, and $x = 2$.

Solution

Here is a diagram showing the region and its center of mass. There are two subregions, as noted. Notice in the solution that we apply the fact that masses and moments are additive.



The center of mass is

$$(\bar{x}, \bar{y}) = \left(\frac{8}{3 + 6\ln 2}, \frac{5}{6 + 12\ln 2}\right) \approx (1.1175, 0.3492)$$

We make use of the result in the next problem (**s560x21**) when analyzing the triangular subregion.

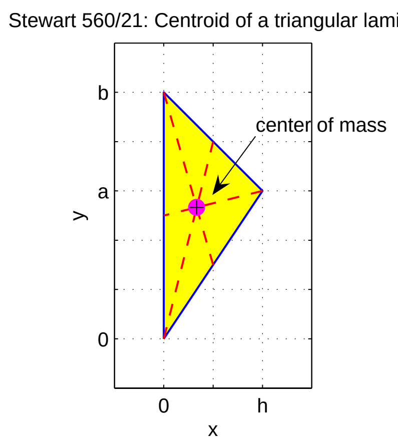
```
%
% Stewart 560/18
%
% Centroid of left triangular region occurs at
% intersection of medians of triangle.
syms d x y
[x1 y1] = solve('y = x/2', 'y = 1-x');
CM1 = [x1 y1]
CM1 =
[ 2/3, 1/3]
m1 = d/2; M1 = m1*CM1;
My1 = M1(1); Mx1 = M1(2);
%
x = linspace(1, 2); y = 1./x;
X = [x 2 0 1]; Y = [y 0 0 1];
fill(X,Y,'y')
grid on; axis equal; hold on
plot(X,Y, 'LineWidth', 2)
axis([-0.5 2.5 -0.5 1.5])
set(gca, 'Xtick', 0:2)
set(gca, 'Ytick', 0:1)
plot([1 1], [0 1], 'r--', 'LineWidth', 2)
%
syms d x; f = 1/x;
m2 = d*int(f, x, 1, 2); pretty(m2)

                                   d log(2)
My2 = d*int(x*f, x, 1, 2); pretty(My1)

                                     1/3 d
Mx2 = d/2*int(f^2, x, 1, 2); pretty(Mx2)

                                     1/4 d
CM2 = simple([My2 Mx1] / m2)
CM2 =
[   1/log(2), 1/6/log(2)]
% Masses and moments are additive.
CM = simple( [My1 + My2  Mx1 + Mx2] / (m1 + m2) );
pretty(CM)

                        [     8                5       ]
                        [------------    -------------]
                        [3 + 6 log(2)    6 + 12 log(2)]
CM = eval(CM)
CM =
    1.1175    0.3492
% %
plot(CM(1), CM(2), 'mo', 'MarkerSize', 8, ...
    'MarkerFaceColor', 'm')
plot(CM(1), CM(2), 'k+', 'MarkerSize', 7)
xlabel('x'); ylabel('y')
title('Stewart 560/18: Centroid of a composite region')
%
echo off; diary off
```

s560x21

Prove that the centroid of a triangular region is located at the point of intersection of the medians of the triangluar boundary.

Solution

Here is a diagram showing the triangular region and its center of mass. You can examine all the analytical work in the code, but the picture tells the story!




```
%
% Stewart 560/21
%
A = 3; B = 5; H = 2;
X = [0 H 0 0]; Y = [0 A B 0];
fill(X,Y,'y')
grid on; axis equal; hold on
plot(X,Y, 'LineWidth', 1)
axis([-1 3 -1 6])
set(gca, 'Xtick', 0:2)
set(gca, 'Ytick', 0:5)
set(gca, 'XtickLabel', {'0', '', 'h'})
set(gca, 'YtickLabel', {'0', '', '', 'a', '', 'b'})
% Medians
plot([0 H/2], [0 (A+B)/2], 'r--', 'LineWidth', 1)
plot([H 0], [A B/2], 'r--', 'LineWidth', 1)
plot([0 H/2], [B A/2], 'r--', 'LineWidth', 1)
%
syms a b d h x
f = (a-b)/h * x + b;   g = a*x/h;
m = d*int(f-g, x, 0, h)
m =
d*(1/2*((a-b)/h-a/h)*h^2+b*h)
My = d*int(x*(f-g), x, 0, h)
My =
d*(1/3*((a-b)/h-a/h)*h^3+1/2*b*h^2)
Mx = d/2*int(f^2 - g^2, x, 0, h)
```

```
Mx =
1/2*d*(1/3*((a-b)^2/h^2-a^2/h^2)*h^3+b*(a-b)*h+b^2*h)
CM = simple([My Mx] / m); pretty(CM)

                                 [1/3 h    1/3 a + 1/3 b]
CM = subs(CM, {a,b,h}, {A,B,H})% CM = eval(CM)
CM =
    0.6667    2.6667
%
plot(CM(1), CM(2), 'mo', 'MarkerSize', 8, ...
    'MarkerFaceColor', 'm')
plot(CM(1), CM(2), 'k+', 'MarkerSize', 7)
xlabel('x'); ylabel('y')
title('Stewart 560/21: Centroid of a triangular lamina')
%
echo off; diary off
```

s560x26

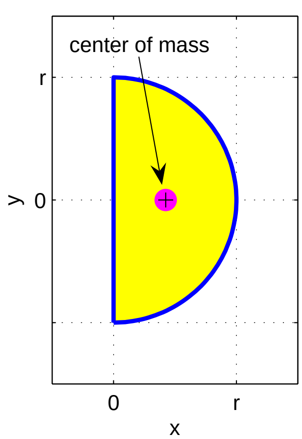
Use the Theorem of Pappus to find the volume of a sphere of radius r.

Solution

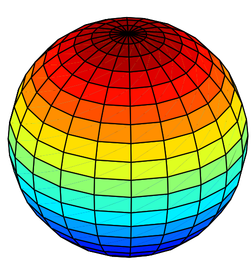
The center of mass of the semicircular region of radius r depicted in the figure at left below is $\left(\frac{4r}{3\pi}, 0\right)$. If we rotate this region about the y-axis, we get a sphere of radius r, depicted on the right. The center of mass traverses the circumference of a circle of radius $R = \frac{4r}{3\pi}$, a distance of $d = C = 2\pi R = 2\pi\left(\frac{4r}{3\pi}\right) = \frac{8r}{3}$. The area of the semicircular region is $A = \frac{1}{2}\pi r^2$. By the Theorem of Pappus, the volume of the sphere is

$$V = Ad = \left(\frac{\pi r^2}{2}\right)\left(\frac{8r}{3}\right) = \frac{4}{3}\pi r^3.$$

Stewart 560/26a: Semicirular region



Stewart 560/26b: Sphere



```
%
% Stewart 560/26
%
syms d r x positive
m = d*int(2*sqrt(r^2 - x^2), x, 0, r); pretty(m)

                                             2
                                1/2 d~ r~   pi
My = d*int(x*2*sqrt(r^2 - x^2), x, 0, r); pretty(My)

                                              3
                                 2/3 d~ r~
Mx = d/2*int(0, x, 0, r); pretty(Mx)

                                       0

CM = [My Mx] / m; pretty(CM)

                                [       r~        ]
                                [4/3 ----       0]
                                [       pi        ]
%
echo off; diary off
```