Spring 2005 Math 152
9 Further Applications of Integration
9.5R Moments and Centers of Mass
Fri, 04/Mar ©2005, Art Belmonte

WARNING!

The treatment given here is the full-blown Calc 3 version. Accordingly, please read the handout "Overview of Vector, Multiple, and Vector-Multiple Integrals" in the Supplements section of our class Web page.

The reason we are able to get away with this level of generality is due to the massive firepower afforded us by MATLAB and the TI-89. That said, we'll also see the enormous amount of hand work going on behind the scenes.

Summary

The point on which a thin plate balances horizontally is called the **center of mass** or **center of gravity** of the plate. Before we deal with this situation (where mass is distributed continuously), let's consider the simpler case of discrete masses.

Discrete masses

Let's consider a system of n discrete masses m_i located at points (x_i, y_i) in the xy-plane, $i = 1, \ldots, n$. For ease of computation, we'll represent the point masses as a row vector $\mathbf{p}$

$$\mathbf{p} = [m_1, m_2, \ldots, m_n]$$

and the points in the plane as rows in a matrix $\mathbf{r}$.

$$\mathbf{r} = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix}$$

- **Total mass**: $m = \sum_{i=1}^{n} m_i$.
  - In MATLAB, **m = sum(p)**.
  - On the TI-89, **rowNorm(p)→m**.
- **Center of mass, CM**: $[\bar{x}, \bar{y}] = \frac{1}{m}(\mathbf{pr})$, a position vector. Note that $\mathbf{pr}$ is *matrix* multiplication. This gives a row vector. Multiplication by $\frac{1}{m}$ is scalar multiplication.
  - In MATLAB, $\mathbf{p} * \mathbf{r}/\mathbf{m}$.
  - On the TI-89, $\mathbf{p} * \mathbf{r}/\mathbf{m}$.

Continuously distributed mass

Now consider a flat plate (or **lamina**) with density ρ that occupies a region in the xy-plane having one of these forms.

$$D = \{(x, y) : f(x) \le y \le g(x), a \le x \le b\}$$
$$D = \{(x, y) : f(y) \le x \le g(y), c \le y \le d\}$$

- **Total mass**: $m = \iint_D \rho \, dA$, realized as either

$$m = \int_a^b \int_{f(x)}^{g(x)} \rho \, dy \, dx$$

  or

$$m = \int_c^d \int_{f(y)}^{g(y)} \rho \, dx \, dy.$$

- **Center of mass, CM**:

$$[\bar{x}, \bar{y}] = \frac{1}{m} \iint_D \rho \, \mathbf{r} \, dA = \frac{1}{m} \iint_D \rho \, [x, y] \, dA,$$

  (where $\mathbf{r} = [x, y]$) realized as either

$$[\bar{x}, \bar{y}] = \frac{1}{m} \int_a^b \int_{f(x)}^{g(x)} \rho \, [x, y] \, dy \, dx$$

  or

$$[\bar{x}, \bar{y}] = \frac{1}{m} \int_c^d \int_{f(y)}^{g(y)} \rho \, [x, y] \, dx \, dy.$$

Advantages of this formulation

1. Notice the similarity between the center of mass in the discrete and continuous cases:

$$[\bar{x}, \bar{y}] = \frac{1}{m}(\mathbf{pr}) \qquad \text{versus} \qquad [\bar{x}, \bar{y}] = \frac{1}{m} \iint_D \rho \, \mathbf{r} \, dA,$$

   respectively.
2. Furthermore, in this formulation the density ρ may be constant *or* variable (a function of x and/or y).
3. Finally, this technique immediately generalizes to three-dimensional space; i.e., the center of mass of a set of discrete masses in space or that of a solid body E in space.
   - In the discrete case, we have $m = \sum_{i=1}^{n} m_i$ with

$$\mathbf{r} = \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ \vdots & \vdots & \vdots \\ x_n & y_n & z_n \end{bmatrix}$$

     and $[\bar{x}, \bar{y}, \bar{z}] = \frac{1}{m}(\mathbf{pr})$, just like before!

- In the continuous case, we have $m = \iiint_E \rho \, dV$ and $[\bar{x}, \bar{y}, \bar{z}] = \dfrac{1}{m} \iiint_E \rho \, \mathbf{r} \, dV$ where $\mathbf{r} = [x, y, z]$. The density ρ may be constant or depend on x, y, and/or z.

Notes

1. Another word for the center of mass of a flat plate (lamina) of uniform (constant) density is the **centroid**. All densities in Section 9.5 are uniform. We'll give an example in this handout that involves a variable density for completeness.
2. The **moment with respect to the x-axis** is defined by $M_x = m\bar{y}$. (Remember, the directed distance from a point to the x-axis is y.)
3. The **moment with respect to the y-axis** is defined by $M_y = m\bar{x}$. (Remember, the directed distance from a point to the y-axis is x.)

Theorem of Pappus

Let R be a plane region lying entirely on one side of a line L in a plane. Let A be the area of R, $P\,(\bar{x}, \bar{y})$ be the center of mass of R, and r be the distance from P to L. Then the volume V of the solid of revolution obtained by rotating R about L is $V = Ad$, where $d = C = 2\pi r$, the circumference of the circle that the center of mass traverses.

Hand / TI-89 Examples

560/4

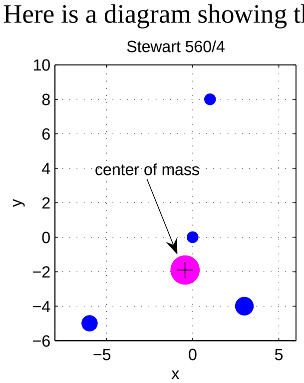
Given masses $m_1 = 3$, $m_2 = 3$, $m_3 = 8$, $m_4 = 6$, located at points

$$P_1\,(0, 0)\,, \quad P_2\,(1, 8)\,, \quad P_3\,(3, -4)\,, \quad P\,(-6, -5)\,,$$

find moments M_x and M_y along with center of mass CM $= (\bar{x}, \bar{y})$.

Solution

Here is a diagram showing the masses and the center of mass.



Let $\mathbf{p} = [3, 3, 8, 6]$ and

$$\mathbf{r} = \begin{bmatrix} 0 & 0 \\ 1 & 8 \\ 3 & -4 \\ -6 & -5 \end{bmatrix}.$$

Then $m = \sum_{i=1}^{n} m_i$ = sum(p) = 20 and the center of mass (CM) is

$$\begin{aligned}
[\bar{x}, \bar{y}] &= \frac{1}{m}\,(\mathbf{pr}) \\
&= \frac{1}{20}\left([3, 3, 8, 6]\begin{bmatrix} 0 & 0 \\ 1 & 8 \\ 3 & -4 \\ -6 & -5 \end{bmatrix}\right) \\
&= \frac{1}{20}\,[0 + 3 + 24 - 36,\ 0 + 24 - 32 - 30] \\
&= \frac{1}{20}\,[-9,\ -38] \\
&= \left[-\frac{9}{20},\ -\frac{19}{10}\right] = [-0.45, -1.90]
\end{aligned}$$

Here matrix multiplication is realized by taking dot products of rows with columns. The moments are as follows.

$$\begin{aligned}
M_x = m\bar{y} &= 20\left(-\tfrac{19}{10}\right) = -38 \\
M_y = m\bar{x} &= 20\left(-\tfrac{9}{20}\right) = -9
\end{aligned}$$

Of course, things are much easier using a TI-89, which has all these operations built-in! (Alternatively use the Matrix Editor to enter **r**.)

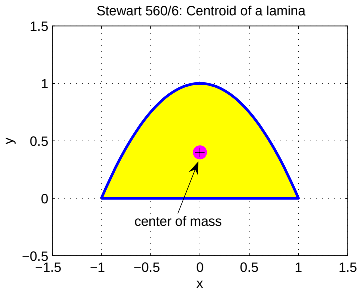
- $[3, 3, 8, 6] \rightarrow \mathbf{p}$
- $[0, 0;\ 1, 8;\ 3, -4;\ -6, -5] \rightarrow \mathbf{r}$
- **rowNorm(p)→m**
- $\mathbf{p} * \mathbf{r} / \mathbf{m}$

560/6

Find the centroid (center of mass of a flat plate of uniform density) of the region bounded by the curves $y = 1 - x^2$ and $y = 0$.

Solution

Here is a diagram showing the region and its center of mass.



First compute the mass

$$m = \iint_D \rho \, dA = \int_{-1}^{1} \int_0^{1-x^2} \rho \, dy \, dx = \frac{4\rho}{3}$$

then compute the center of mass. (Here ρ is constant.)

$$\begin{aligned} [\bar{x}, \bar{y}] &= \frac{1}{m} \iint_D \rho \, [x, y] \, dA \\ &= \frac{1}{4\rho/3} \int_{-1}^{1} \int_0^{1-x^2} \rho \, [x, y] \, dy \, dx \\ &= \left[0, \frac{2}{5}\right] \end{aligned}$$

These two steps are easy to do on a TI-89, especially if you use the **Muint** (multiple integral) menu in the TAMUCALC package. (You'll find ρ on the **Calc** menu.)

- $\int(\int(\rho, y, 0, 1 - x \wedge 2), x, -1, 1) \to$ **m**
- $\int(\int(\rho * [x, y], y, 0, 1 - x \wedge 2), x, -1, 1)/$**m**

The second command is even easier than you think. Just change the $\to$ (from the store operation) in the entry line to $/$ for division and post-multiply ρ by $[x, y]$.

"Fine, Wise Guy. But what is *really* going on?!" A world 'o' hurt, my friend. Indeed, a veritable veil of tears. . . Observe. (Recall that constants slide out across integration.)

$$\begin{aligned} m &= \iint_D \rho \, dA = \int_{-1}^{1} \int_0^{1-x^2} \rho \, dy \, dx \\ &= \rho \int_{-1}^{1} y \Big|_0^{1-x^2} dx \\ &= \rho \int_{-1}^{1} 1 - x^2 \, dx \\ &= 2\rho \int_0^1 1 - x^2 \, dx \quad \text{[via symmetry]} \\ &= 2\rho \left(x - \tfrac{1}{3}x^3\right) \Big|_0^1 = \frac{4\rho}{3} - 0 = \frac{4\rho}{3} \end{aligned}$$

That wasn't too bad. But we're just getting started. . .

$$\begin{aligned} [\bar{x}, \bar{y}] &= \frac{1}{m} \iint_D \rho \, [x, y] \, dA \\ &= \frac{1}{4\rho/3} \int_{-1}^{1} \int_0^{1-x^2} \rho \, [x, y] \, dy \, dx \\ &= \frac{3}{4} \int_{-1}^{1} \int_0^{1-x^2} [x, y] \, dy \, dx \\ &= \frac{3}{4} \left[\int_{-1}^{1} \int_0^{1-x^2} x \, dy \, dx, \int_{-1}^{1} \int_0^{1-x^2} y \, dy \, dx \right] \\ &= \frac{3}{4} \left[\int_{-1}^{1} xy \Big|_{y=0}^{y=1-x^2} dx, \int_{-1}^{1} \tfrac{1}{2} y^2 \Big|_0^{1-x^2} dx \right] \\ &= \frac{3}{4} \left[\int_{-1}^{1} x - x^3 \, dx, \int_{-1}^{1} \tfrac{1}{2} \left(1 - x^2\right)^2 dx \right] \\ &= \frac{3}{4} \left[0, \int_0^1 1 - 2x^2 + x^4 \, dx \right] \quad \text{[via symmetry]} \\ &= \frac{3}{4} \left[0, \left(x - \tfrac{2}{3}x^3 + \tfrac{1}{5}x\right) \Big|_0^1 \right] \\ &= \frac{3}{4} \left[0, \frac{6}{5} - \frac{2}{3} \right] = \frac{3}{4} \left[0, \frac{18 - 10}{15} \right] \\ &= \frac{3}{4} \left[0, \frac{8}{15} \right] = \left[0, \frac{2}{5} \right] \end{aligned}$$

The TI-89 performed these computations for $[\bar{x}, \bar{y}]$ *in 1 second.* This is why we use machines!

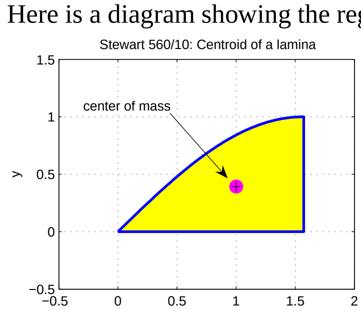
In the rest of this handout we'll only give the summary results of the integrals. *Do them on your TI-89!* On a common exam, just compute $\bar{x}$, the x-coordinate of the center of mass. See the 9.5X handout. $\boxed{m = \iint_D \rho \, dA \text{ and } \bar{x} = \frac{1}{m} \iint_D \rho x \, dA}$

560/10

Find the centroid (center of mass of a flat plate of uniform density) of the region bounded by the curves $y = \sin x$, $y = 0$, $x = 0$, and $x = \frac{\pi}{2}$.

Solution

Here is a diagram showing the region and its center of mass.



First compute the mass

$$m = \iint_D \rho \, dA = \int_0^{\pi/2} \int_0^{\sin x} \rho \, dy \, dx = \rho$$

then compute the center of mass. (Here ρ is constant.)

$$\begin{aligned}[\bar{x}, \bar{y}] &= \frac{1}{m} \iint_D \rho \, [x, y] \, dA \\ &= \frac{1}{\rho} \int_0^{\pi/2} \int_0^{\sin x} \rho \, [x, y] \, dy \, dx \\ &= \left[1, \frac{\pi}{8}\right] \approx [1.00, 0.39]\end{aligned}$$

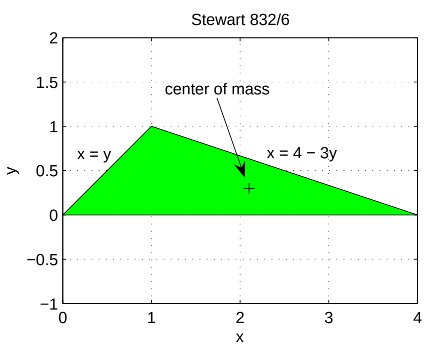
- $\int \left(\int \left(\rho, y, 0, \sin(x) \right), x, 0, \pi/2 \right) \to$**m**
- $\int \left(\int \left(\rho * [x, y], y, 0, \sin(x) \right), x, 0, \pi/2 \right) /$**m**

832/6 [from Section 13.6 in Math 253 (Calc 3)]

Find the mass and center of mass of the lamina (flat plate) that occupies the triangular region D in the xy-plane with vertices $(0, 0)$, $(1, 1)$, and $(4, 0)$ and has *variable* density $\rho = x$.

Solution

Here is a plot showing the region and its center of mass (+).



With our general formulation, the fact that the density is variable presents no difficulty whatsoever! First compute the mass

$$m = \iint_D \rho \, dA = \int_0^1 \int_y^{4-3y} x \, dx \, dy = \frac{10}{3}$$

then compute the center of mass.

$$\begin{aligned}[\bar{x}, \bar{y}] &= \frac{1}{m} \iint_D \rho \, [x, y] \, dA \\ &= \frac{1}{10/3} \int_0^1 \int_y^{4-3y} x[x, y] \, dx \, dy \\ &= \left[\frac{21}{10}, \frac{3}{10}\right] = [2.1, 0.3]\end{aligned}$$

- $x \to \rho$
- $\int \left(\int \left(\rho, x, y, 4 - 3 * y \right), y, 0, 1 \right) \to$**m**
- $\int \left(\int \left(\rho * [x, y], x, y, 4 - 3 * y \right), y, 0, 1 \right) /$**m**
- **DelVar** ρ (Note that **NewProb** clears only single letter Roman identifiers—not Greek ones!)

MATLAB Examples

We'll repeat the TI-89 examples using MATLAB. The syntax for the integrals involved is almost identical! Just replace $\int$ with **int**. For brevity, we'll type **p** for the density instead of **rho** (for ρ).

s560x04 [560/4 revisited]

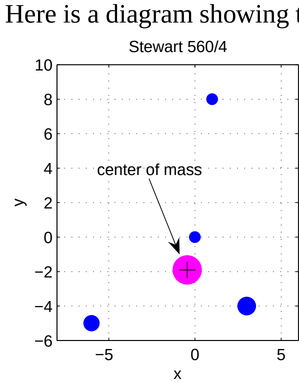
Given masses $m_1 = 3$, $m_2 = 3$, $m_3 = 8$, $m_4 = 6$, located at points

$$P_1\,(0, 0)\,, \quad P_2\,(1, 8)\,, \quad P_3\,(3, -4)\,, \quad P\,(-6, -5)\,,$$

find moments M_x and M_y along with center of mass $\text{CM} = (\bar{x}, \bar{y})$.

Solution

Here is a diagram showing the masses and the center of mass.



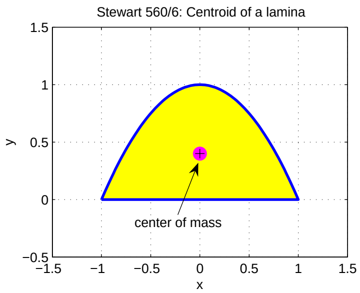

```
%
% Stewart 560/4
%
p = [3 3 8 6]                 % masses
p =
     3     3     8     6
r = [0 0; 1 8; 3 -4; -6 -5] % points (x: col 1, y: col 2)
r =
     0     0
     1     8
     3    -4
    -6    -5
m = sum(p)                    % total mass
m =
    20
CM = p*r/m                    % center of mass
CM =
   -0.4500   -1.9000
%                             %   via matrix multiplication
echo off; diary off
```

s560x06 [560/6 revisited]

Find the centroid (center of mass of a flat plate of uniform density) of the region bounded by the curves $y = 1 - x^2$ and $y = 0$.

Solution

Here is a diagram showing the region and its center of mass.



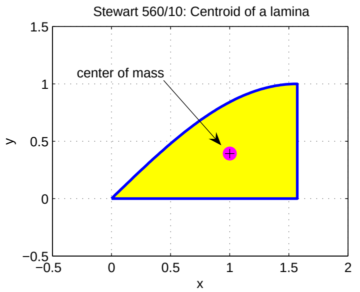

```
%
% Stewart 560/6
%
syms p x y
m = int(int(p, y, 0, 1-x^2), x, -1, 1);
pretty(m)

                                     4/3 p
CM = 1/m * int(int(p*[x y], y, 0, 1-x^2), x, -1, 1);
pretty(CM)

                                  [0    2/5]
```

s560x10 [560/10 revisited]

Find the centroid (center of mass of a flat plate of uniform density) of the region bounded by the curves $y = \sin x$, $y = 0$, $x = 0$, and $x = \frac{\pi}{2}$.

Solution

Here is a diagram showing the region and its center of mass.



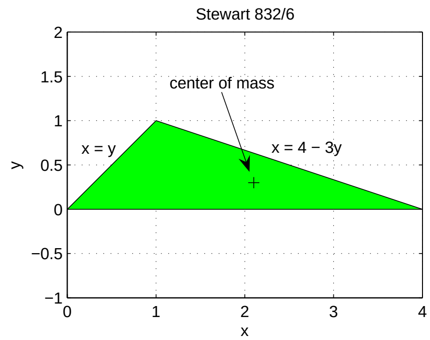

```
%
% Stewart 560/10
%
syms p x y
m = int(int(p, y, 0, sin(x)), x, 0, pi/2);
pretty(m)

                                      p
CM = 1/m * int(int(p*[x y], y, 0, sin(x)), x, 0, pi/2);
pretty(CM)

                                 [1    1/8 pi]
CM = eval(CM)
CM =

    1.0000    0.3927
```

s832x06

Find the mass and center of mass of the lamina (flat plate) that occupies the triangular region D in the xy-plane with vertices $(0,0)$, $(1,1)$, and $(4,0)$ and has *variable* density $\rho = x$.

Solution

Here is a plot showing the region and its center of mass (+).



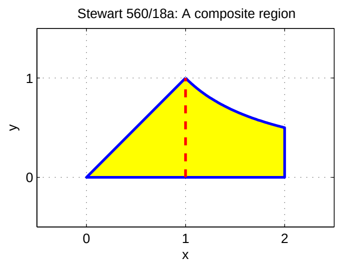

```
%
% Stewart 832/6
%
syms x y
p = x;
m = int(int(p, x, y, 4-3*y), y, 0, 1);
pretty(m)

                                     10/3
CM = 1/m * int(int(p*[x y], x, y, 4-3*y), y, 0, 1);
pretty(CM)

                                 [21        ]
                                 [--    3/10]
                                 [10        ]
```

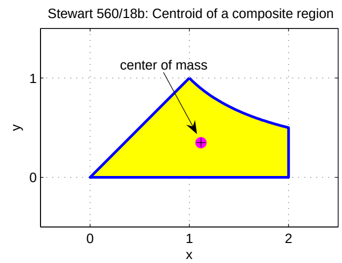
s560x18

Find the centroid of the region bounded by $y = x$, $y = 0$, $y = 1/x$, and $x = 2$.

Solution

Here is a diagram showing the region and its center of mass. There are two subregions, so we must split the integrals involved.





The center of mass is

$$(\bar{x}, \bar{y}) = \left(\frac{8}{3+6\ln 2}, \frac{5}{6+12\ln 2}\right) \approx (1.1175, 0.3492)$$

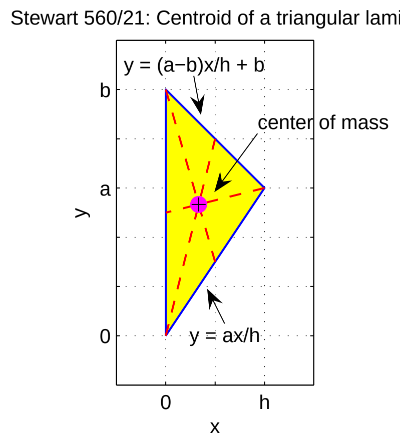
```
%
% Stewart 560/18
%
syms p x y
m = int(int(p, y, 0, x), x, 0, 1) + ...
    int(int(p, y, 0, 1/x), x, 1, 2);
pretty(m)

                               1/2 p + log(2) p
CM = 1/m * ...
    ( int(int(p*[x y], y, 0, x), x, 0, 1) ...
    + int(int(p*[x y], y, 0, 1/x), x, 1, 2) );
CM = simple(CM)
CM =
[  8/(3+6*log(2)), 5/(6+12*log(2))]
pretty(CM)

                        [      8              5       ]
                        [------------    -------------]
                        [3 + 6 log(2)    6 + 12 log(2)]
CM = eval(CM)
CM =
    1.1175    0.3492
%
echo off; diary off
```

s560x21

Prove that the centroid of a triangular region is located at the point of intersection of the medians of the triangluar boundary.

Solution

Here is a diagram showing the triangular region and its center of mass. You can examine all the analytical work in the code, but the picture tells the story!




```
%
% Stewart 560/21
%
syms a b h p x y
%
% Vertices
V1 = [0 0]; V2 = [h a]; V3 = [0 b];
% Slanted sides
L1 = cline2pt(V1, V2); pretty(L1)

                                     a x
                                 y = ---
                                      h
L2 = cline2pt(V3, V2); pretty(L2)

                                (a - b) x
                            y = --------- + b
                                    h
% Median lines
M1 = cline2pt(V1, (V2+V3)/2); pretty(M1)

                                  x (a + b)
                              y = ---------
                                      h
M2 = cline2pt(V2, (V1+V3)/2); pretty(M2)

                                2 a x - x b + b h
                        y = 1/2 -----------------
                                        h
M3 = cline2pt(V3, (V1+V2)/2); pretty(M3)

                              (-2 b + a) x
                          y = ------------ + b
                                   h
% Common intersection of medians
I1 = solve(M1, M2, x, y);
I2 = solve(M1, M3, x, y);
I3 = solve(M2, M3, x, y);
I = [I1.x I1.y; I2.x I2.y; I3.x I3.y]; pretty(I)

                           [1/3 h    1/3 a + 1/3 b]
                           [                      ]
                           [1/3 h    1/3 a + 1/3 b]
                           [                      ]
                           [1/3 h    1/3 a + 1/3 b]
%
m = int(int(p, y, a*x/h, (a-b)*x/h + b), x, 0, h);
pretty(m)

                              /a - b       \  2
                        1/2 p |----- - a/h| h   + p b h
                              \  h         /
CM = 1/m * ...
    int(int(p*[x y], y, a*x/h, (a-b)*x/h + b), x, 0, h);
CM = simple(CM);
pretty(CM) % Centroid is at intersection of medians.

                           [1/3 h    1/3 a + 1/3 b]
```

s560x26

Use the Theorem of Pappus to find the volume of a sphere of radius r.

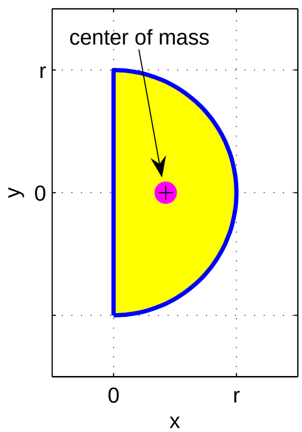
Solution

The center of mass of the semicircular region of radius r depicted in the figure at left below is $\left(\frac{4r}{3\pi}, 0\right)$. If we rotate this region about the y-axis, we get a sphere of radius r, depicted on the right.

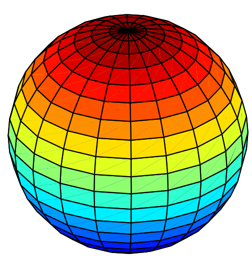
The center of mass traverses the circumference of a circle of radius $R = \frac{4r}{3\pi}$, a distance of $d = C = 2\pi R = 2\pi\left(\frac{4r}{3\pi}\right) = \frac{8r}{3}$. The area of the semicircular region is $A = \frac{1}{2}\pi r^2$. By the Theorem of Pappus, the volume of the sphere is

$$V = Ad = \left(\frac{\pi r^2}{2}\right)\left(\frac{8r}{3}\right) = \frac{4}{3}\pi r^3.$$

Stewart 560/26a: Semicirular region



Stewart 560/26b: Sphere



```
%
% Stewart 560/26
%
syms r x positive
syms p y
m = int(int(p, x, 0, sqrt(r^2 - y^2)), y, -r, r);
pretty(m)

                                      2
                                1/2 r~  pi p

CM = 1/m * ...
    int(int(p*[x y], x, 0, sqrt(r^2 - y^2)), y, -r, r);
pretty(CM)

                               [      r~      ]
                               [4/3 ----     0]
                               [      pi      ]
%
echo off; diary off
```