Spring 2005 Math 152
9 Further Applications of Integration
9.5X *x*-coordinate of Center of Mass
Wed, 08/Mar ©2006, Art Belmonte

Summary

(For the full treatment, see 9.5R, the regular lecture. Or view a streamlined version in 9.5E that uses machine power exclusively.) To obtain just $\bar{x}$, the x-coordinate of the center of mass when the density is constant, compute these two integrals.

$$\begin{aligned} \text{area } A &= \int_a^b f(x) - g(x)\, dx \\ \bar{x} &= \frac{1}{A}\int_a^b x\,(f(x) - g(x))\, dx \end{aligned}$$

Note that this does *not* give you the center of mass, just half of the information needed. That said, this is all that will be required on Common Exam 2. (On the final, of course, you will compute the full center of mass with machine power.)

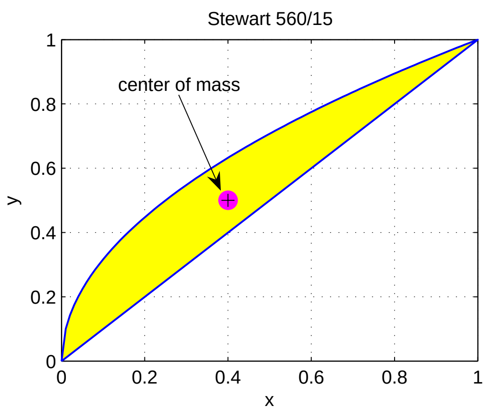
Hand Examples

560/15

Find $\bar{x}$, the x-coordinate of the centroid (center of mass of a flat plate of uniform density) of the region bounded by the curves $y = \sqrt{x}$ and $y = x$.

Solution

Here is a diagram showing the region and its center of mass.



First compute the area A.

$$\begin{aligned} A &= \int_a^b f(x) - g(x)\, dx \\ &= \int_0^1 x^{1/2} - x\, dx \\ &= \left(\tfrac{2}{3}x^{3/2} - \tfrac{1}{2}x^2\right)\Big|_0^1 \\ &= \frac{1}{6} \end{aligned}$$

Now compute $\bar{x}$, the x-coordinate of the center of mass.

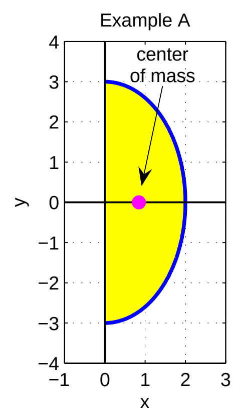
$$\begin{aligned} \bar{x} &= \frac{1}{A}\int_a^b x\,(f(x) - g(x))\, dx \\ &= \frac{1}{1/6}\int_0^1 x\left(x^{1/2} - x\right) dx \\ &= 6\int_0^1 x^{3/2} - x^2\, dx \\ &= 6\left(\tfrac{2}{5}x^{5/2} - \tfrac{1}{3}x^3\right)\Big|_0^1 \\ &= 6\left(\frac{6}{15} - \frac{5}{15}\right) = \frac{6}{15} = \frac{2}{5} \end{aligned}$$

Example A

Find $\bar{x}$, the x-coordinate of the centroid (center of mass of a flat plate of uniform density) of the region in the right half-plane bounded by the y-axis and the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$.

Solution

Here is a diagram showing the region and its center of mass.



First compute the area A.

$$\begin{aligned} A &= \int_a^b f(x) - g(x)\, dx \\ &= \int_0^2 \sqrt{9\left(1 - \frac{x^2}{4}\right)} - \left(-\sqrt{9\left(1 - \frac{x^2}{4}\right)}\right) dx \\ &= \int_0^2 2\sqrt{9\left(1 - \frac{x^2}{4}\right)}\, dx \\ &= 3\pi \end{aligned}$$

Now compute $\bar{x}$, the x-coordinate of the center of mass.

$$\begin{aligned}
\bar{x} &= \frac{1}{A}\int_a^b x\,(f(x) - g(x))\,dx \\
&= \frac{1}{3\pi}\int_0^2 x\left(\sqrt{9\left(1-\frac{x^2}{4}\right)} - \left(-\sqrt{9\left(1-\frac{x^2}{4}\right)}\right)\right)dx \\
&= \frac{1}{3\pi}\int_0^2 2x\sqrt{9\left(1-\frac{x^2}{4}\right)}\,dx \\
&= \frac{8}{3\pi} \approx 0.85
\end{aligned}$$