Spring 2006 Math 152/STEPS
Numerical Solution of Second-Order Differential Equations
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Summary

We may convert the second-order differential equation

$$y'' = g\left(t, y, y'\right)$$

(in so-called *normal* form) along with the initial conditions

$$y\left(t_0\right) = y_0, \quad y'\left(t_0\right) = y_1$$

into a *system* of two first-order equations as follows.

Let $x_1 = y$ and $x_2 = y'$. Then we have the pair of first-order differential equations

$$\begin{aligned} x_1' &= y' = x_2 \\ x_2' &= y'' = g\left(t, y, y'\right) = g\left(t, x_1, x_2\right) \end{aligned}$$

and the following initial conditions on x_1 and x_2.

$$\begin{aligned} x_1\left(t_0\right) &= y\left(t_0\right) = y_0 \\ x_2\left(t_0\right) &= y'\left(t_0\right) = y_1 \end{aligned}$$

In other words, we have the *vector* differential equation

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} x_2 \\ g\left(t, x_1, x_2\right) \end{bmatrix}$$

and *vector* initial condition

$$\begin{bmatrix} x_1\left(t_0\right) \\ x_2\left(t_0\right) \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}.$$

More concisely, we have $\mathbf{x}' = \mathbf{g}\left(t, \mathbf{x}\right)$ along with $\mathbf{x}\left(t_0\right) = \mathbf{x}_0$. Here $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\mathbf{x}' = \begin{bmatrix} x_1' \\ x_2' \end{bmatrix}$, $\mathbf{g}\left(t, \mathbf{x}\right) = \begin{bmatrix} x_2 \\ g\left(t, x_1, x_2\right) \end{bmatrix}$, and $\mathbf{x}_0 = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}$.

REMARK With this framework it is easy to convert an n^{th}-order differential equation together with n initial conditions into a system of n first-order differential equations and initial conditions. That is the beauty of the vector formulation. For now, we'll just deal with second-order equations.

MATLAB Examples

Once a problem has been formulated as a system of first-order equations and initial conditions, we can bring MATLAB's **ode45** routine to bear on numerically solving the problem and then graphing it. The advantage of this method is that is allows very general types of problems to be solved.

- linear or nonlinear
- homogeneous or nonhomogeneous
- constant or variable coefficient

Example A

Solve the initial value problem

$$y'' + 16y = 8\cos 4t, \quad y\left(0\right) = 0, \quad y'\left(0\right) = 0.$$

numercialy with MATLAB's **ode45**, then graph the solution for $0 \le t \le 4\pi$.

Solution

1. First write the differential equation in normal form.

$$y'' = 8\cos 4t - 16y$$

2. Let $x_1 = y$ and $x_2 = y'$. Then we have the pair of first-order differential equations

$$\begin{aligned} x_1' &= y' = x_2 \\ x_2' &= y'' = 8\cos 4t - 16y = 8\cos 4t - 16x_1 \end{aligned}$$

   and the following initial conditions on x_1 and x_2.

$$\begin{aligned} x_1\left(0\right) &= y\left(0\right) = 0 \\ x_2\left(0\right) &= y'\left(0\right) = 0 \end{aligned}$$

3. Formulated as a vector differential equation, we have

$$\mathbf{x}' = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} x_2 \\ 8\cos 4t - 16x_1 \end{bmatrix}$$

   and

$$\mathbf{x}\left(0\right) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\left(0\right) = \begin{bmatrix} x_1\left(0\right) \\ x_2\left(0\right) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

4. We write a script M-file driver that controls execution. Here are some notes.
   - The variable **tspan** is the time span; i.e., $0 \le t \le 4\pi$.
   - The variable **x0** specifies the vector initial condition as a *column* vector.
   - The reference **@f** is called a "function handle." It points to the function M-file that contains derivative information gleaned from the vector differential equation.
   - The variable **t** contains the actual time values. Note that **ode45** is a *variable*-step method as opposed to a constant-step method.
   - The variable **x** contains the numerical solution values for x_1 in column 1 and x_2 in column 2.
   - Since $y = x_1$, we extract our y-values from the first column of **x**.
   - Plot t on the horizontal axis and y on the vertical axis.
   - The dashed curves are called "envelopes."
   - The mechanical phenomenon exhibited is called "resonance."

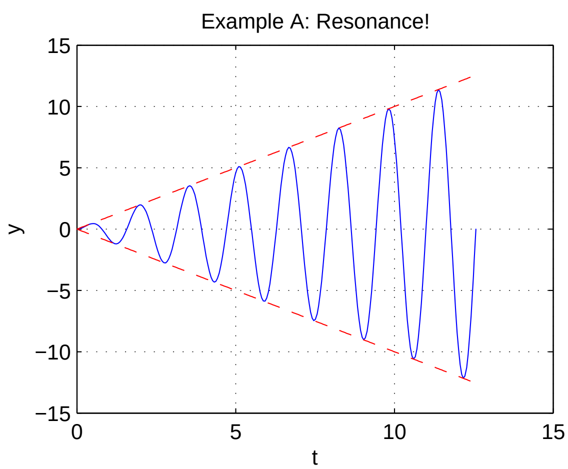
```
%--------------------
% THIS IS THE DRIVER, ExA.m
%
% Example A: Resonance!
%
tspan = [0 4*pi]; x0 = [0 0]';
[t x] = ode45(@f, tspan, x0);
y = x(:,1);
plot(t,y); grid on; hold on
plot(t,t,'r--')
plot(t,-t,'r--')
xlabel('t'); ylabel('y')
title('Example A: Resonance!')
%
echo off; diary off
```

5. We must also write a function M-file that contains derivative information gleaned from the vector differential equation. Here are some notes.

   - The derivative information is returned in **xp**, which stands for **x**-prime; i.e., the derivative of **x**. It is a 2×1 *column* vector.
   - We preallocate the vector **xp** by dimensioning it and filling it with zeros.
   - Then we specify its components.

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= 8\cos 4t - 16x_1 \end{aligned}$$

```
%--------------------
function xp = f(t,x)
xp = zeros(2,1); % column vector!
xp(1) = x(2);
xp(2) = 8*cos(4*t) - 16*x(1);
```



Example B

Solve the initial value problem

$$4y'' + 1024y = 50\cos 15t, \quad y(0) = 0, \quad y'(0) = 0.$$

numercialy with MATLAB's **ode45**, then graph the solution for $0 \le t \le 6\pi$.

Solution

1. First write the differential equation in normal form.

$$y'' = \frac{1}{4}(50\cos 4t - 1024y)$$

2. Let $x_1 = y$ and $x_2 = y'$. Then we have the pair of first-order differential equations

$$\begin{aligned} x_1' &= y' = x_2 \\ x_2' &= y'' = \frac{1}{4}(50\cos 4t - 1024y) = \frac{1}{4}(50\cos 4t - 1024x_1) \end{aligned}$$

and the following initial conditions on x_1 and x_2.

$$\begin{aligned} x_1(0) &= y(0) = 0 \\ x_2(0) &= y'(0) = 0 \end{aligned}$$

3. Formulated as a vector differential equation, we have

$$\mathbf{x}' = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} x_2 \\ \frac{1}{4}(50\cos 4t - 1024x_1) \end{bmatrix}$$

and

$$\mathbf{x}(0) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

4. We write a script M-file driver that controls execution. Here are some notes.

   - The variable **tspan** is the time span; i.e., $0 \le t \le 6\pi$.
   - The variable **x0** specifies the vector initial condition as a *column* vector.
   - The reference **@f** is called a "function handle." It points to the function M-file that contains derivative information gleaned from the vector differential equation.
   - The variable **t** contains the actual time values. Note that **ode45** is a *variable*-step method as opposed to a constant-step method.
   - The variable **x** contains the numerical solution values for x_1 in column 1 and x_2 in column 2.
   - Since $y = x_1$, we extract our y-values from the first column of **x**.
   - Plot t on the horizontal axis and y on the vertical axis.
   - The dashed curves are called "envelopes."
   - The mechanical phenomenon exhibited is called "beats."

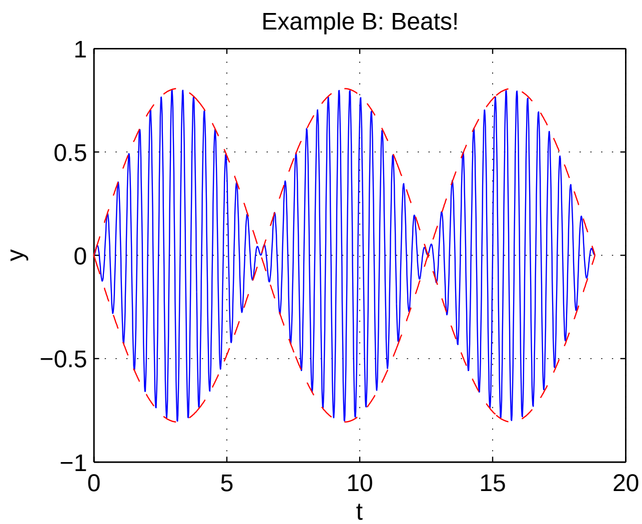
```
%--------------------
% THIS IS THE DRIVER, ExB.m
%
% Example B: Beats!
%
tspan = [0 6*pi]; x0 = [0 0]';
[t x] = ode45(@f, tspan, x0);
y = x(:,1);
plot(t,y); grid on; hold on
y = 25/31*sin(t/2);
plot(t,y,'r--')
plot(t,-y,'r--')
xlabel('t'); ylabel('y')
title('Example B: Beats!')
%
echo off; diary off
```

5. We must also write a function M-file that contains derivative information gleaned from the vector differential equation. Here are some notes.

- The derivative information is returned in **xp**, which stands for **x**-prime; i.e., the derivative of **x**. It is a 2×1 *column* vector.
- We preallocate the vector **xp** by dimensioning it and filling it with zeros.
- Then we specify its components.

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= \frac{1}{4}(50\cos 4t - 1024x_1) \end{aligned}$$

```
%--------------------
function xp = f(t,x)
xp = zeros(2,1); % column vector!
xp(1) = x(2);
xp(2) = (50*cos(15*t) - 1024*x(1)) / 4;
```



Example C

Fluid Ejection In the design of a sewage treatment plant, the following initial value problem arises:

$$60 - H = 77.7H'' + 19.42\left(H'\right)^2, \quad H(0) = 0, \quad H'(0) = 0,$$

where H is the level of fluid in an ejection chamber and t is the time in seconds. Solve this problem numerically using MATLAB's **ode45**, then graph $H(t)$ for $0 \le t \le 5$.

Solution

1. First write the differential equation in normal form.

$$H'' = \frac{1}{77.7}\left(60 - H - 19.42\left(H'\right)^2\right)$$

2. Let $x_1 = H$ and $x_2 = H'$. Then we have the pair of first-order differential equations

$$\begin{aligned} x_1' &= H' = x_2 \\ x_2' &= H'' = \frac{1}{77.7}\left(60 - H - 19.42\left(H'\right)^2\right) \\ &= \frac{1}{77.7}\left(60 - x_1 - 19.42x_2^2\right) \end{aligned}$$

and the following initial conditions on x_1 and x_2.

$$\begin{aligned} x_1(0) &= H(0) = 0 \\ x_2(0) &= H'(0) = 0 \end{aligned}$$

3. Formulated as a vector differential equation, we have

$$\mathbf{x}' = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} x_2 \\ \frac{1}{77.7}(60 - x_1 - 19.42x_2^2) \end{bmatrix}$$

and

$$\mathbf{x}(0) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

4. We write a script M-file driver that controls execution. Here are some notes.

- The variable **tspan** is the time span; i.e., $0 \le t \le 5$.
- The variable **x0** specifies the vector initial condition as a *column* vector.
- The reference **@f** is called a "function handle." It points to the function M-file that contains derivative information gleaned from the vector differential equation.
- The variable **t** contains the actual time values. Note that **ode45** is a *variable*-step method as opposed to a constant-step method.
- The variable **x** contains the numerical solution values for x_1 in column 1 and x_2 in column 2.
- Since $H = x_1$, we extract our H-values from the first column of **x**.
- Plot t on the horizontal axis and H on the vertical axis.

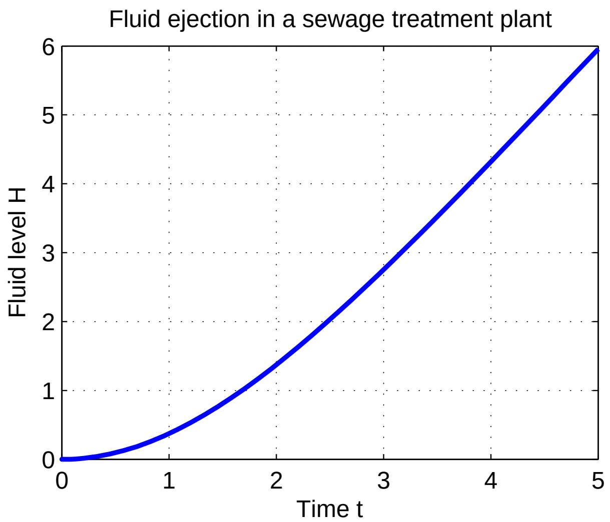
```
%--------------------
% THIS IS THE DRIVER, sewage.m
%
% Sewage treatment plant problem
%
tspan = [0 5]; x0 = [0 0]';
[t x] = ode45(@f,tspan,x0);
H = x(:,1);
plot(t,H, 'LineWidth', 2); grid on
xlabel('Time t');
ylabel('Fluid level H')
title('Fluid ejection in a sewage treatment plant')
%
echo off; diary off
```

5. We must also write a function M-file that contains derivative information gleaned from the vector differential equation. Here are some notes.

- The derivative information is returned in **xp**, which stands for **x**-prime; i.e., the derivative of **x**. It is a 2×1 *column* vector.
- We preallocate the vector **xp** by dimensioning it and filling it with zeros.
- Then we specify its components.

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= \frac{1}{77.7}\left(60 - x_1 - 19.42x_2^2\right) \end{aligned}$$

```
%----------
function xp = f(t,x);
xp = zeros(2,1); % column vector!
xp(1) = x(2);
xp(2) = (60 - x(1) - 19.42.*x(2).^2) / 77.7;
```



Notes

1. The exact solution to Example A is

$$y = t \sin 4t.$$

This may be computing using either the Method of Undetermined Coefficients or Variation of Parameters.

2. The exact solution to Example B is

$$y = \frac{25}{62}\left(\cos 15t - \cos 16t\right).$$

This may be computing using either the Method of Undetermined Coefficients or Variation of Parameters. Furthermore, via a trigonometric identity we can more readily understand the phenomenon of beats. Now

$$\sin\left(\frac{\alpha+\beta}{2}t\right)\sin\left(\frac{\alpha-\beta}{2}t\right) = -\frac{1}{2}\left(\cos\alpha t - \cos\beta t\right)$$

whence

$$\begin{aligned} \cos\alpha t - \cos\beta t &= -2\sin\left(\frac{\alpha+\beta}{2}t\right)\sin\left(\frac{\alpha-\beta}{2}t\right) \\ &= 2\sin\left(\frac{\beta+\alpha}{2}t\right)\sin\left(\frac{\beta-\alpha}{2}t\right). \end{aligned}$$

Therefore

$$y = \frac{25}{62}\left(\cos 15t - \cos 16t\right) = \frac{25}{31}\sin\left(\frac{31t}{2}\right)\sin\left(\frac{t}{2}\right),$$

the product of a rapidly varying sine function with a slowly varying sine function. Indeed, $y = A\sin\left(\frac{31}{2}t\right)$, where $A = \frac{25}{31}\sin\left(\frac{1}{2}t\right)$ is a time-varying amplitude. The dashed red envelope curves in the plot are precisely $\pm A$.

3. No exact solution is attainable to Example C. Yet we are able to *numerically* solve and graph it in precisely the same manner as for examples A and B!