

Spring 2006 Math 152/STEPS

2nd-Order Linear Nonhomogeneous Constant Coefficient Differential Equations: Variation of Parameters

Wed, 08/Mar ©2006, Art Belmonte

Summary

Let $L[y] = y'' + by' + cy = f$ be a second-order linear nonhomogeneous constant coefficient differential equation in *standard linear form* (SLF). This means that the coefficient of y'' is $a = 1$.

Recall that if y_p is a particular solution of this nonhomogeneous equation and y_h is a general solution to the associated homogeneous equation $L[y] = 0$, then a general solution of the nonhomogeneous equation is given by $y = y_p + y_h$.

Note that f may be of a more general form than that of the restricted set of functions that the Method of Undetermined Coefficients requires. (Even further, one may actually have variable coefficients that depend on the independent variable instead of just having the constants b and c .)

Variation of Parameters (VOP)

The following method produces a **general solution** of the nonhomogeneous equation. *First make sure the differential equation in standard linear form!*

1. Obtain a fundamental set of solutions $\mathbf{y}_f = [y_1, y_2]$, a row vector, of $L[y] = 0$, the corresponding homogeneous equation. Given a column vector of constants, $\mathbf{c} = [c_1; c_2]$, form a general solution of this homogeneous equation, $y_h = \mathbf{y}_f \mathbf{c} = c_1 y_1 + c_2 y_2$.
2. Form the column vector $\mathbf{b} = [0; f]$, whose first entry is zero and whose second entry is the right-hand side of the nonhomogeneous equation in SLF. Compute the Wronskian matrix of \mathbf{y}_f , $\mathbf{W} = \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}$. Solve $\mathbf{W}\mathbf{v}_p = \mathbf{b}$ for \mathbf{v}_p (\mathbf{v} prime), a column vector: $\mathbf{v}_p = \mathbf{W}^{-1}\mathbf{b} = \mathbf{W}\backslash\mathbf{b}$.
3. Integrate \mathbf{v}_p to obtain $\mathbf{v} = [v_1; v_2]$, a column vector. (Don't worry about constants of integration.)
4. A **particular solution** of the nonhomogeneous equation is $y_p = \mathbf{y}_f \mathbf{v} = v_1 y_1 + v_2 y_2$.
5. A **general solution** of the nonhomogeneous equation is $y = y_p + y_h = (v_1 y_1 + v_2 y_2) + (c_1 y_1 + c_2 y_2)$.

Use MATLAB or a TI-89 for your computations!

The operations involved in the variation of parameters procedure are tailor-made for MATLAB or the TI-89: matrix multiplication, solving linear systems, vector integration, etc.

MATLAB Examples

To facilitate computation of the Wronskian matrix, I wrote a function M-file named `wron`. Type "`help wron`" at a MATLAB prompt to learn about it. If you'd like to see the (short) code, type "`type wron`" at a MATLAB prompt.

Example A

Find a general solution of $y'' + 4y = \sec 2t$. (Why can't we use the method of undetermined coefficients here?)

Solution

We paraphrase the steps. Computations are done with MATLAB or a TI-89.

1. The corresponding homogeneous equation is $y'' + 4y = 0$. Its characteristic equation, $r^2 + 4 = 0$, has roots $r = \pm 2i$. A fundamental set of solutions is given by the row vector $\mathbf{y}_f = [\cos 2t, \sin 2t]$. A general solution of the homogeneous equation is $y_h = c_1 \cos 2t + c_2 \sin 2t$.
2. Noting that the DE is in SLF, form the column vector $\mathbf{b} = [0; \sec 2t]$ and the Wronskian matrix for \mathbf{y}_f : $\mathbf{W} = \begin{bmatrix} \cos 2t & \sin 2t \\ -2 \sin 2t & 2 \cos 2t \end{bmatrix}$. Solve the linear system $\mathbf{W}\mathbf{v}_p = \mathbf{b}$ to obtain $\mathbf{v}_p = \mathbf{W}\backslash\mathbf{b} = [-\frac{1}{2} \tan 2t; \frac{1}{2}]$.
3. Integrate \mathbf{v}_p to obtain $\mathbf{v} = \int \mathbf{v}_p dt = [\frac{1}{4} \ln(\cos 2t); \frac{1}{2}t]$. (Don't worry about constants of integration.)
4. A particular solution of the nonhomogeneous equation is given by the matrix or dot product

$$y_p = \mathbf{y}_f \mathbf{v} = \frac{1}{4} \cos 2t \ln(\cos 2t) + \frac{1}{2} t \sin 2t.$$

5. A general solution is $y = y_p + y_h$ or

$$y = \frac{1}{4} \cos 2t \ln(\cos 2t) + \frac{1}{2} t \sin 2t + c_1 \cos 2t + c_2 \sin 2t.$$

Here is a short MATLAB input file.

```
%
delete ExA.txt; diary ExA.txt
clear; clc; close all; echo on
%
% Example A
%
syms c1 c2 r t
p = poly2sym([1 0 4], r); pretty(p)
r = solve(p)
yf = [cos(2*t) sin(2*t)];
c = [c1; c2];
yh = yf*c;
%
W = wron(yf, t)
b = [0; sec(2*t)]
vp = simple(W\b)
v = int(vp, t)
%
yp = yf*v;
Y = yp + yh; pretty(Y)
%
echo off; diary off
```

Here is the corresponding diary file.

```

%
% Example A
%
syms c1 c2 r t
p = poly2sym([1 0 4], r); pretty(p)

                2
                r  + 4

r = solve(p)

r =

[ 2*i]
[-2*i]

yf = [cos(2*t) sin(2*t)];
c = [c1; c2];
yh = yf*c;
%
W = wron(yf, t)

W =

[ cos(2*t), sin(2*t)]
[-2*sin(2*t), 2*cos(2*t)]

b = [0; sec(2*t)]

b =

[ 0]
[ sec(2*t)]

vp = simple(W\b)

vp =

[-1/2*sin(2*t)/cos(2*t)]
[ 1/2]

v = int(vp, t)

v =

[ 1/4*log(cos(2*t))]
[ 1/2*t]

%
yp = yf*v;
Y = yp + yh; pretty(Y)

1/4 cos(2 t) log(cos(2 t)) + 1/2 sin(2 t) t
+ cos(2 t) c1 + sin(2 t) c2
%
echo off; diary off

```

Example B

Find a general solution of $x'' - 4x' + 4x = e^{2t}$. (Can we use the method of undetermined coefficients here?)

Solution

The MATLAB input file (here and in every VOP problem) is structurally the same. Here is a diary file.

```

%
% Example B
%
syms c1 c2 r t
p = poly2sym([1 -4 4], r); pretty(p)

                2
                r  - 4 r + 4

r = solve(p)

r =

```

```

[ 2]
[ 2]

yf = [exp(2*t) t*exp(2*t)];
c = [c1; c2];
yh = yf*c;
%
W = wron(yf, t)

W =

[ exp(2*t), t*exp(2*t)]
[ 2*exp(2*t), exp(2*t)+2*t*exp(2*t)]

b = [0; exp(2*t)]

b =

[ 0]
[ exp(2*t)]

vp = simple(W\b)

vp =

[-t]
[ 1]

v = int(vp, t)

v =

[-1/2*t^2]
[ t]

%
yp = yf*v;
Y = yp + yh; pretty(Y)

1/2 exp(2 t) t  + exp(2 t) c1 + t exp(2 t) c2
%
echo off; diary off

```