Spring 2005 Math 152
10 Infinite Sequences and Series
10.2 Series
Wed, 23/Mar ©2005, Art Belmonte

Summary

Concepts

- Given a sequence $\{a_k\}$, we form an **infinite series**

  $$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \cdots$$

  by adding up the terms of the sequence in order. The variable k is called the **index** of the series. Other commonly used indices are i, j, m, and n.

- Associated with the series $\sum_{k=1}^{\infty} a_k$ is its corresponding **sequence of partial sums** $\{s_n\}$ where

  $$s_n = \sum_{k=1}^{n} a_k.$$

  If $\{s_n\}$ converges to $s = \lim_{n\to\infty} s_n$, then the series $\sum_{k=1}^{\infty} a_k$ is said to **converge** to s or be convergent with **sum** s and we write $\sum_{k=1}^{\infty} a_k = s$. If $\{s_n\}$ does *not* converge, then the series $\sum_{k=1}^{\infty} a_k$ is said to **diverge** or be **divergent**.

- A **telescoping sum** is one which, through cancellation of terms, collapses to the sum of two (or a few) terms. [See 593/22 in the Hand Examples.]

Theorems

Sum Laws for Series Suppose that $\sum_{n=1}^{\infty} a_n = A$ and $\sum_{n=1}^{\infty} b_n = B$ are convergent series with the stated sums and c is a constant. Then the following infinite sums exist and have the stated values.

- $\sum_{n=1}^{\infty} (a_n + b_n) = A + B$
- $\sum_{n=1}^{\infty} (a_n - b_n) = A - B$
- $\sum_{n=1}^{\infty} c a_n = cA$

A *necessary* condition for series convergence If the series $\sum_{n=1}^{\infty} a_n$ converges, then the sequence of terms $\{a_n\}$ must converge to 0. [**WARNING**: This is *not* a *sufficient* condition. Indeed, the **harmonic series** $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges to ∞ even though the sequence of terms $\frac{1}{n}$ conveges to 0. See 590/Example 7 in the Hand Examples.]

Test for Divergence If $\lim_{n\to\infty} a_n \neq 0$ (i.e., the limit either exists and is nonzero or it does not exist), then the corresponding series $\sum_{n=1}^{\infty} a_n$ diverges in light of the previous paragraph.

Geometric Series Theorem (GST) Let r be a fixed real number. The **geometric series** $\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$ converges for $|r| < 1$:

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \qquad |r| < 1.$$

If $|r| \geq 1$, the series diverges.

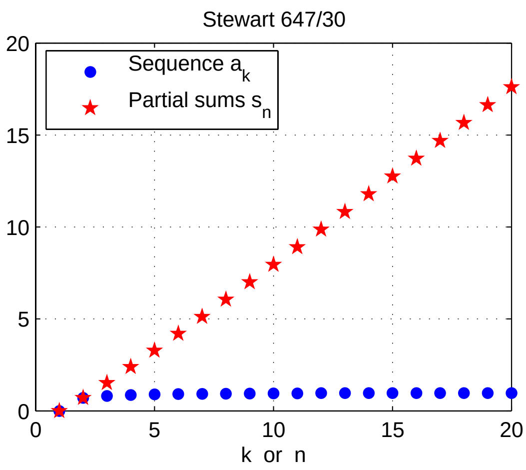
Hand Examples

647/30

Determine whether the series $\sum_{n=1}^{\infty} \sqrt{\frac{n-1}{n}}$ is convergent.

Solution

Because $a_n = \sqrt{\frac{n-1}{n}} = \sqrt{1 - \frac{1}{n}} \to 1$ as $n \to \infty$, the series *diverges* by the Test for Divergence. Here is an illustrative plot showing the terms of the series together with its partial sums.



593/12

Determine whether the series $1 - \frac{3}{2} + \frac{9}{4} - \frac{27}{8} + \cdots$ converges. If so, find its limit.

Solution

This is a geometric series with ratio $r = -\frac{3}{2}$. Since $|r| = \frac{3}{2} > 1$, the series *diverges* by the Geometric Series Test.

593/16

Determine whether the series $\sum_{n=1}^{\infty} \frac{4^{n+1}}{5^n}$ is convergent.

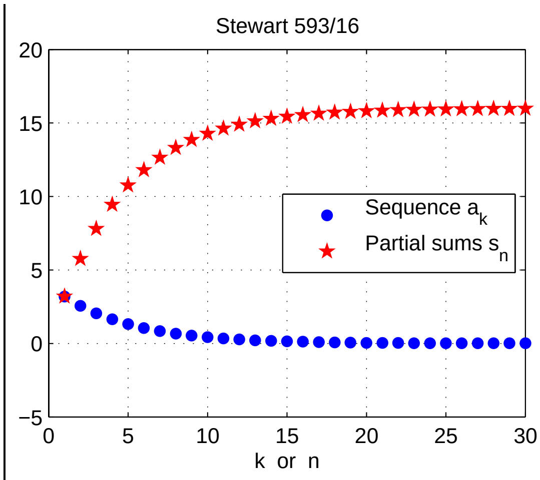
If so, find its sum.

Solution

This is actually a *convergent* geometric series that sums to 16.

$$\sum_{n=1}^{\infty} \left(\frac{4^2}{5}\right)\left(\frac{4}{5}\right)^{n-1} = \sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} = \frac{16/5}{1/5} = 16$$

Here is an illustrative plot showing the terms of the series together with its partial sums.



593/22

Determine whether the series $\sum_{k=1}^{\infty} \frac{1}{4k^2-1}$ is convergent.

If so, find its sum.

Solution

- Let's split the k^{th} term into partial fractions.

$$\begin{aligned} \frac{1}{4k^2-1} = \frac{1}{(2k-1)(2k+1)} &= \frac{A}{2k-1} + \frac{B}{2k+1} \\ 1 &= A(2k+1) + B(2k-1) \\ 0k + 1 &= (2A+2B)k + (A-B) \end{aligned}$$

  Now $2A + 2B = 0$ and $A - B = 1$, so $A = \frac{1}{2}$ and $B = -\frac{1}{2}$.

- The series is thus equivalent to $\sum_{k=1}^{\infty} \frac{1}{2}\left(\frac{1}{2k-1} - \frac{1}{2k+1}\right)$, which is a telescoping sum. Let's look at its partial sums.

$$\begin{aligned} s_1 &= \tfrac{1}{2}\left(\tfrac{1}{1} - \tfrac{1}{3}\right) \\ s_2 &= \tfrac{1}{2}\left(\left(\tfrac{1}{1} - \tfrac{1}{3}\right) + \left(\tfrac{1}{3} - \tfrac{1}{5}\right)\right) = \tfrac{1}{2}\left(1 - \tfrac{1}{5}\right) \\ s_3 &= \tfrac{1}{2}\left(\left(\tfrac{1}{1} - \tfrac{1}{3}\right) + \left(\tfrac{1}{3} - \tfrac{1}{5}\right) + \left(\tfrac{1}{5} - \tfrac{1}{7}\right)\right) = \tfrac{1}{2}\left(1 - \tfrac{1}{7}\right) \\ &\vdots \\ s_n &= \frac{1}{2}\left(1 - \frac{1}{2n+1}\right) \to \frac{1}{2} \text{ as } n \to \infty \end{aligned}$$

- Therefore $\sum_{k=1}^{\infty} \frac{1}{4k^2-1}$ is *convergent* and its sum is $1/2$.

590/Example 7 [harmonic series]

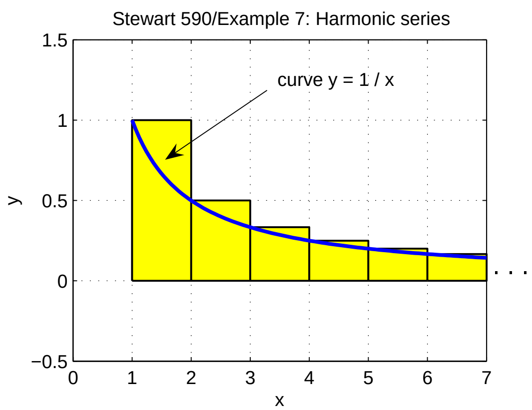
Show that the **harmonic series** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Solution

Look at the plot below. The sum of the areas of the yellow rectangular regions is at least as great as the area under the curve $y = 1/x$, $1 \le x < \infty$. This implies

$$\sum_{n=1}^{\infty} \frac{1}{n} \ge \int_1^{\infty} \frac{1}{x}\,dx = \lim_{t\to\infty} (\ln x)\Big|_1^t = \lim_{t\to\infty} \ln t = \infty.$$

In other words, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ *diverges* to ∞.



593/27

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \tan^{-1} n$ converges.

Solution

Since $a_n = \tan^{-1} n \to \frac{\pi}{2} \ne 0$ as $n \to \infty$, the series *diverges* by the Test for Divergence.

593/28

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right)$ converges.

Solution

Since $\ln\left(\dfrac{n}{n+1}\right) = \ln n - \ln(n+1)$, the series is another telescoping sum. Let's look at the partial sums of the series.

$$\begin{aligned}
s_1 &= \ln 1 - \ln 2 = -\ln 2 \\
s_2 &= (\ln 1 - \ln 2) + (\ln 2 - \ln 3) = -\ln 3 \\
s_3 &= (\ln 1 - \ln 2) + (\ln 2 - \ln 3) + (\ln 3 - \ln 4) = -\ln 4 \\
&\vdots \\
s_n &= -\ln(n+1) \to -\infty \text{ as } n \to \infty.
\end{aligned}$$

Therefore, the series $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right)$ *diverges* to $-\infty$.

MATLAB Examples

MATLAB is probably the best tool I've encountered for use with sequences. Here are some relevant commands.

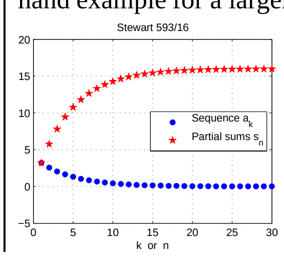
- element-by-element operations: .* ./ .^
- built-in math functions: **sin**, **cos**, **exp**, **log**, **sqrt**, etc.
- **plot**
- **cumsum**: does all cumulative sums at once!
- **symsum**: Symbolic sums? You bet!

s593x16 [593/16 revisited]

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{4^{n+1}}{5^n}$ is convergent. If so, find its sum.

Solution

Here we plot the terms in the series together with the partial sums. We also compute the full infinite sum via **symsum**. (Look at the hand example for a larger plot.)



```
%
% Stewart 593/16
%
% Sequence a
%
n = 1:30; a = 4.^(n+1) ./ 5.^n;
plot(n,a,'bo', 'MarkerFaceColor', 'b', ...
    'MarkerSize', 5)
xlabel('k  or  n')
title('Stewart 593/16')
grid on; hold on
%
% Partial sums s
%
s = cumsum(a);
plot(n,s,'rp', 'MarkerFaceColor', 'r', ...
    'MarkerSize', 7)
legend('Sequence a_k', 'Partial sums s_n', ...
    'Location', 'East')
axis([0 30 -5 20])
%
% Sum of infinite series.
%
syms n
S = symsum(4^(n+1) / 5^n, n, 1, inf);
pretty(S)

                                                          16
%
echo off; diary off
```

593/32

Express the repeating decimal number $4.\overline{1570}$ as a ratio of integers.

Solution

By hand, we split the number into the sum of an integer and a geometric series.

$$\begin{aligned} 4.\overline{1570} &= 4+\sum_{n=1}^{\infty}\left(\left(\frac{1570}{10^4}\right)\left(\frac{1}{10^4}\right)^{n-1}\right) \\ &= 4+\frac{a}{1-r}=4+\frac{\frac{1570}{10^4}}{1-\frac{1}{10^4}} \\ &= 4+\frac{1570}{9999}=\frac{41,566}{9999} \end{aligned}$$

Right... How about if we just squash this bug with **symsum**? That's better!

```
%
% Stewart 593/32
%
syms n
S = 4 + symsum((1570/10^4) * (1/10^4)^(n-1), n, 1, inf)
S =
41566/9999
%
echo off; diary off
```

s594x42a

A certain ball has the property that each time it falls from a height h onto a hard, level surface, it rebounds to a height rh, where $0<r<1$ is fixed. Suppose the ball is dropped from an intial height of H meters. Assuming that the ball continues to bounce indefinitely, find the total distance that it travels.

Solution

The ball initially falls H meters,

- then bounces up $(H)\,r$ meters and falls down Hr meters,
- then bounces up $(Hr)\,r=Hr^2$meters and falls down Hr^2 meters,
- then bounces up $\left(Hr^2\right)r=Hr^3$meters and falls down Hr^3 meters, etc.

Therefore, the total distance traveled is given by

$$\begin{aligned} &= H+2Hr+2Hr^2+2Hr^3+\cdots \\ &= -H+2H+2Hr+2Hr^2+2Hr^3+\cdots \\ &= -H+\sum_{n=1}^{\infty}\left(2Hr^{n-1}\right) \\ \text{[via GST]}\rightarrow &= -H+\frac{2H}{1-r}=\frac{2H-H+Hr}{1-r}=\left(\frac{1+r}{1-r}\right)H \end{aligned}$$

We check our work with **symsum**.

```
%
% Stewart 594/42a
%
syms H n r
D = -H + symsum(2*H*r^(n-1), n, 1, inf);
pretty(D)

                                           H
                                  -H - 2 -----
                                         r - 1
hand_answer = H * (1+r) / (1-r); pretty(hand_answer)

                                      H (r + 1)
                                      ---------
                                        1 - r
no_diff = simple(D - hand_answer)
no_diff =
0
%
echo off; diary off
```