Spring 2005 Math 152
10 Infinite Sequences and Series
10.3 *Positive Series:* The Integral and Comparison Tests; Estimating Sums
Mon, 28/Mar ©2005, Art Belmonte

Summary

In this section, all series have *positive* terms!

We'll use $\sum a_n$ as a shorthand for $\sum_{n=1}^{\infty} a_n$.

The Integral Test and Estimating Infinite Sums

The Integral Test Let f be a continuous, positive, [ultimately] decreasing function on $[N, \infty)$ and let $a_n = f(n)$. (Here N is a positive integer.) Then the series $\sum a_n$ converges if and only if $\int_N^{\infty} f(x)\,dx$ converges. That is, the series and integral exhibit the same behavior: *either both converge or both diverge*.

The *p*-series The series $\sum \frac{1}{n^p}$ converges for $p > 1$ and diverges for $p \le 1$.

The Remainder Estimate for the Integral Test Let $\sum a_n$ be a convergent series whose sum is s and let $s_n = \sum_{k=1}^{n} a_k$ be its n^{th} partial sum. The n^{th} **remainder** is $R_n = s - s_n = \sum_{k=n+1}^{\infty} a_k$.

It satisfies $\int_{n+1}^{\infty} f(x)\,dx \le R_n \le \int_n^{\infty} f(x)\,dx$ for $n \ge N$.

Bounds on the infinite sum *s* Adding s_n to the last inequality gives $s_n + \int_{n+1}^{\infty} f(x)\,dx \le s \le s_n + \int_n^{\infty} f(x)\,dx$ for $n \ge N$.

Comparison Tests

If $\sum b_n$ converges and $a_n \le b_n$ for $n \ge N$, then $\sum a_n$ converges.

If $\sum b_n$ diverges and $a_n \ge b_n$ for $n \ge N$, then $\sum a_n$ diverges.

Limit Comparison Test

Let $\lim_{n\to\infty} (a_n/b_n) = L \ge 0$. There are three conclusive cases.

- If $L = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.
- If $L = c > 0$, then $\sum a_n$ and $\sum b_n$ exhibit similar behavior: *either both converge or both diverge*. (This case is typical.)
- If $L = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

Hand Examples

603/4

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms and $\sum b_n$ is known to diverge.

(a) If $a_n > b_n$ for all n, what can you say about $\sum a_n$ and why?

(b) If $a_n < b_n$ for all n, what can you save about $\sum a_n$ and why?

Solution

(a) The series $\sum a_n$ *diverges* by the Comparison Test.

(b) No conclusion can be made about the series $\sum a_n$.
It may converge or it may diverge. Indeed,

- if $a_n = \frac{1}{2} b_n$, then the series $\sum a_n$ *diverges* by the Limit Comparison Test since $\lim (a_n/b_n) = \frac{1}{2} > 0$ and $\sum b_n$ diverges;
- whereas if $a_n = \min\left\{\left(\frac{1}{2}\right)^n, b_n\right\}$, then the Geometric Series Theorem implies $\sum a_n \le \sum \left(\frac{1}{2}\right)^n = 1$ and thus the series $\sum a_n$ *converges* by the Comparison Test.

604/6

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2}{n\sqrt{n}} + \frac{3}{n^3}\right)$ converges.

Solution

Let $a_n = 1/n^{3/2}$ and $b_n = 1/n^3$. Since $A = \sum a_n$ and $B = \sum b_n$ are convergent p-series ($p = \frac{3}{2} > 1$ and $p = 3 > 1$, respectively), our series, $\sum (2a_n + 3b_n)$, *converges* to $2A + 3B$ by the sum laws for series in Section 13.2.

604/8

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges.

Solution

This series *converges* by the Integral Test since

$$\int_1^{\infty} \frac{1}{x^2+1}\,dx = \lim_{t\to\infty} \left(\tan^{-1} x \Big|_1^t\right) = \lim_{t\to\infty} \left(\tan^{-1} t - \frac{\pi}{4}\right) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}.$$

604/12

Determine whether the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1}$ is convergent.

Solution

Since $\dfrac{1}{\sqrt{n}-1} > \dfrac{1}{\sqrt{n}} = \dfrac{1}{n^{1/2}}$ for $n \geq 2$ and $\sum \frac{1}{n^{1/2}}$ is a divergent p-series ($p = \frac{1}{2} \leq 1$), the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1}$ *diverges* by the Comparison Test.

604/18

Determine whether the series $\sum a_n = \displaystyle\sum_{n=1}^{\infty} \frac{1+2^n}{1+3^n}$ is convergent.

Solution

- *Essentially*, since $a_n \sim \frac{2^n}{3^n} = \left(\frac{2}{3}\right)^n = b_n$ and the (geometric) series $\sum b_n$ converges, our series $\sum a_n$ also *converges* by the Limit Comparison Theorem. [Here the symbol $\sim$ means "is similar to" or "behaves like" for large values of n.]
- *More formally*, let $b_n = \left(\frac{2}{3}\right)^n$. Then as $n \to \infty$ we have

$$\frac{a_n}{b_n} = \frac{(1+2^n)\,3^n}{(1+3^n)\,2^n} = \frac{3^n+6^n}{2^n+6^n} = \frac{\left(\frac{1}{2}\right)^n + 1}{\left(\frac{1}{3}\right)^n + 1} \to 1 > 0.$$

  Since $\sum b_n$ converges by the Geometric Series Theorem ($|r| = \frac{2}{3} < 1$) and the behavior of $\sum a_n$ and $\sum b_n$ are similar by the Limit Comparison Theorem, we conclude that the series $\sum a_n$ *converges*.

MATLAB Examples

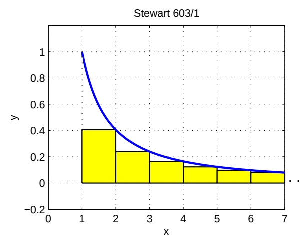
s603x01

Draw a picture to show that $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^{1.3}} \leq \int_1^{\infty} \frac{1}{x^{1.3}}\,dx$.

What can you conclude about the series?

Solution

The sum $s = \sum \frac{1}{n^{1.3}}$ of the areas of the rectangles is less than the area $\int_1^{\infty} \frac{1}{x^{1.3}}\,dx = \frac{10}{3}$ under the curve. Therefore, this p-series ($p = 1.3 > 1$) converges.




```
%
% Stewart 603/1
%
Ln = rsum(@f, 1, 7, 6, 1);
title('Stewart 603/1')
I = int(1 / x^1.3, x, 1, inf)
I =
10/3
%
echo off; diary off
```

604/29

Use the first 10 terms of the series $\sum a_n = \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4+n^2}$ to approximate its sum s. Estimate the error.

Solution

- An estimate of the sum is $s \approx s_{10} = \sum_{n=1}^{10} a_n \approx 0.5680$.
- The error $R_{10} = s - s_{10}$ satisfies

$$R_{10} \leq \int_{10}^{\infty} \frac{1}{x^4+x^2}\,dx = \frac{1}{10} - \frac{\pi}{2} + \tan^{-1} 10 \approx 3.3135 \times 10^{-4}.$$

- A slightly looser upper bound on the error (*but one that's easy to compute by hand*) is obtained as follows.

$$\begin{aligned} R_{10} \leq \int_{10}^{\infty} \frac{1}{x^4+x^2}\,dx &\leq \int_{10}^{\infty} \frac{1}{x^4}\,dx \\ &= \lim_{t\to\infty} \left(-\tfrac{1}{3}x^{-3}\right)\Big|_{10}^{t} \\ &= \lim_{t\to\infty} \left(\frac{1}{3000} - \frac{1}{3t^3}\right) \\ &= \frac{1}{3000} \approx 3.3\bar{3} \times 10^{-4} \end{aligned}$$

```
%
% Stewart 604/29
%
n = 1:10; a = 1 ./ (n.^4 + n.^2); s_10 = sum(a)
s_10 =
    0.5680
%
syms n x
U_R10 = int(1 / (x^4 + x^2), x, 10, inf); pretty(U_R10)

                          - 1/2 pi + atan(10) + 1/10
% (via partial fractions)
eval(U_R10)
ans =
   3.3135e-04
echo off; diary off
```