Spring 2005 Math 152
10 Infinite Sequences and Series
10.4 Other Convergence Tests
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Summary

Alternating series and tests regarding them

An **alternating series** $\sum (-1)^n b_n$ is a series whose terms are (ultimately) alternately positive and negative ($b_n > 0$ for $n \geq N$).

The Alternating Series Test (AST) If $b_n \geq b_{n+1} > 0$ for $n \geq N$ and $b_n \to 0$ as $n \to \infty$, then $\sum (-1)^n b_n$ converges. (We write $b_n \downarrow 0$ to signify that the b_n decrease to 0 in the limit.)

The Alternating Series Estimation Test (ASET) Suppose that $\sum (-1)^k b_k$ is an alternating series that converges to s. Let s_n be its n^{th} partial sum. Then the n^{th} remainder $R_n = s - s_n$ satisfies $|R_n| \leq b_{n+1}$. In other words, the magnitude of the error does not exceed that of the first neglected term.

Absolute and conditional convergence

- An **absolutely convergent** series $\sum a_n$ is one whose series of absolute values $\sum |a_n|$ converges.
- A series is **conditionally convergent** if $\sum a_n$ converges but $\sum |a_n|$ diverges.
- **THEOREM**: If a series $\sum a_n$ is absolutely convergent, then it is convergent.

The Ratio Test

Let $\lim_{n\to\infty} \left|a_{n+1}/a_n\right| = L \geq 0$. There are three possibilities.

- If $L < 1$, then $\sum a_n$ *converges absolutely*.
- If $L > 1$ or $L = \infty$, then $\sum a_n$ *diverges*.
- If $L = 1$, the test is *inconclusive*.

The Root Test

Let $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L \geq 0$. There are three possibilities.

- If $L < 1$, then $\sum a_n$ *converges absolutely*.
- If $L > 1$ or $L = \infty$, then $\sum a_n$ *diverges*.
- If $L = 1$, the test is *inconclusive*.

Boosting the power of the Root Test

Generalized Fun Fact (GFF) Let $p(n) = \sum_{k=0}^{m} c_k n^k$ be a polynomial in n with $c_m > 0$. Then $\lim_{n\to\infty} \sqrt[n]{p(n)} = 1$.

- For example, $\lim_{n\to\infty} \sqrt[n]{9n^3 - 15n^2 + 11n - 8} = 1$.
  "In the fullness of time, all the nastiness goes away."

Stirling's Formula As $n \to \infty$, we have $n! \sim \sqrt{2n\pi}\,(n/e)^n$. In other words, $\lim_{n\to\infty} \dfrac{\sqrt{2n\pi}\,(n/e)^n}{n!} = 1$.

Hand Examples

611/4

Test the convergence of the series

$$\frac{1}{\ln 2} - \frac{1}{\ln 3} + \frac{1}{\ln 4} - \frac{1}{\ln 5} + \frac{1}{\ln 6} - \cdots$$

Solution

Since $b_n = \dfrac{1}{\ln(n+1)} \downarrow 0$, the given alternating series *converges* by the Alternating Series Test (AST).

611/6

Test the convergence of the series $\sum a_n = \sum_{n=1}^{\infty} (-1)^n \dfrac{n^2}{n^2+1}$.

Solution

Since $\lim a_n \neq 0$, the series *diverges* by the Divergence Test from Section 10.2. (The sequence of terms a_n diverges by oscillation.)

611/8

Test the convergence of the series $\sum_{n=1}^{\infty} (-1)^n \dfrac{\ln n}{n}$.

Solution

Now $\lim_{n\to\infty} \dfrac{\ln n}{n} \overset{\text{L'H}}{=} \lim_{n\to\infty} \dfrac{1/n}{1} = 0$ and $\frac{d}{dx}\left(\frac{\ln x}{x}\right) = \frac{1-\ln x}{x^2} < 0$ for $x \geq 3$. Hence $\dfrac{\ln n}{n} \downarrow 0$. So the series *converges* by the AST.

611/20

Determine whether the series $\sum_{n=1}^{\infty} \dfrac{(-3)^n}{n!}$ is absolutely convergent, conditionally convergent, or divergent.

Solution

The series *converges absolutely* by the Ratio Test.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left(\frac{3^{n+1}}{(n+1)!}\cdot\frac{n!}{3^n}\right) = \lim_{n\to\infty}\frac{3}{n+1} = 0 < 1$$

611/22

Determine whether the series $\sum a_n = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}\sqrt{n}}{n+1}$ is absolutely convergent, conditionally convergent, or divergent.

Solution

The series *converges conditionally* since it converges by the AST but $\sum |a_n|$ diverges by the Limit Comparison Test. Here are the details regarding these two assertions.

- First, $\dfrac{\sqrt{n}}{n+1} = \dfrac{1}{\sqrt{n}+\frac{1}{\sqrt{n}}} \to 0$ as $n \to \infty$. Next,

  $$\frac{d}{dx}\left(\frac{x^{1/2}}{x+1}\right) = \frac{(x+1)\left(\frac{1}{2}x^{-1/2}\right) - x^{1/2}(1)}{(x+1)^2} = \frac{1-x}{2\sqrt{x}\,(x+1)^2} < 0$$

  for $x > 1$. So $\frac{\sqrt{n}}{n+1} \downarrow 0$ and the series *converges* by the AST.
- However, $\sum |a_n| = \sum \frac{\sqrt{n}}{n+1} \sim \sum b_n = \sum \frac{1}{n^{1/2}}$, whence $\sum |a_n|$ *diverges* by the Limit Comparison Test since $\sum \frac{1}{n^{1/2}}$ is a divergent p-series ($p = \frac{1}{2} < 1$). [Observe that as $n \to \infty$, we have $\dfrac{|a_n|}{b_n} = \dfrac{n}{n+1} = \dfrac{1}{1+\frac{1}{n}} \to 1 > 0$.]

Example A

Determine whether the series $\sum_{n=1}^{\infty} e^{-n}n!$ is absolutely convergent, conditionally convergent, or divergent.

Solution

The series *diverges* by the Ratio Test. Note that $e^{-n}n! = \dfrac{n!}{e^n}$. So

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left(\frac{(n+1)!}{e^{n+1}}\cdot\frac{e^n}{n!}\right) = \lim_{n\to\infty}\frac{n+1}{e} = \infty$$

611/24

Determine whether the series $\sum a_n = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}5^{n-1}}{(n+1)^2\,4^{n+2}}$ is absolutely convergent, conditionally convergent, or divergent.

Solution

The series *diverges* by the Root Test (& GFF) or via the Ratio Test.

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\frac{5/4}{\sqrt[n]{80\,(n+1)^2}} = \frac{5/4}{1} > 1$$

611/26

Determine whether the series $\sum a_n = \sum_{n=1}^{\infty}\frac{\cos(n\pi/6)}{n\sqrt{n}}$ is absolutely convergent, conditionally convergent, or divergent.

Solution

Now $\sum |a_n| \le \sum \frac{1}{n^{3/2}}$, a convergent p-series ($p = \frac{3}{2} > 1$). Therefore $\sum |a_n|$ converges by the Comparison Test from Section 10.3. Hence $\sum_{n=1}^{\infty}\frac{\cos(n\pi/6)}{n\sqrt{n}}$ is *absolutely convergent*.

Example B

Determine whether the series $\sum a_n = \sum_{n=2}^{\infty}\frac{(-1)^n}{n\ln n}$ is absolutely convergent, conditionally convergent, or divergent.

Solution

The series $\sum a_n$ converges by the AST, but $\sum |a_n|$ diverges by the Integral Test. Hence $\sum a_n$ is *conditionally convergent*. Here are details regarding these assertions.

- Clearly, $b_n = |a_n| = \frac{1}{n\ln n} \downarrow 0$. So the alternating series $\sum a_n$ converges by the AST.
- However, $\sum |a_n|$ diverges by the Integral Test since

  $$\begin{aligned}\int_2^{\infty}\frac{1}{x\ln x}\,dx &= \lim_{t\to\infty}\left(\ln(\ln x)\Big|_2^t\right)\\ &= \lim_{t\to\infty}(\ln(\ln t) - \ln(\ln 2)) = \infty.\end{aligned}$$
- Therefore, $\sum a_n$ is *conditionally convergent*.

Example C

Determine whether the series $\sum a_n = \sum_{n=2}^{\infty}\frac{(-1)^n}{(\tan^{-1}n)^n}$ is absolutely convergent, conditionally convergent, or divergent.

Solution

The series converges absolutely via the Root Test.

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\frac{1}{\tan^{-1}n} = \frac{1}{\pi/2} = \frac{2}{\pi} < 1$$

611/30

The series $\sum a_n$ is **recursively defined** by

$$a_1 = 1, \qquad a_{n+1} = \frac{(2+\cos n)\, a_n}{\sqrt{n}} \text{ for } n \geq 1.$$

Determine whether it converges or diverges.

Solution

Indeed, the series *converges absolutely* by the Ratio Test.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\dfrac{(2+\cos n)\, a_n}{\sqrt{n}}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{2+\cos n}{\sqrt{n}} \right| = 0 < 1$$

MATLAB Examples

s611x12

How many terms of the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4}$ do we need to add in order to find the exact sum s to within an error of $\epsilon = 10^{-3}$?

Solution

The Alternating Series Estimation Theorem (ASET) says that the magnitude of the error does not exceed that of the first neglected term of the series.

$$\begin{aligned} |R_N| \leq \frac{1}{(N+1)^4} &\overset{\text{want}}{\leq} \epsilon = 10^{-3} = \frac{1}{10^3} \\ 10^3 &\leq (N+1)^4 \\ N &\geq 10^{3/4} - 1 \approx 4.62 \end{aligned}$$

So choose $N = 5 = \text{ceil}\,(4.62)$ terms. (Recall that indices are positive integers.) In this case we have $s_4 \approx 0.9475$; that is, $s = 0.9475 \pm 0.001$.

```
%
% Stewart 611/12
%
N = ceil( 10^(3/4) - 1 )
N =
     5
n = 1:N; a = (-1).^(n+1) ./ n.^4;
s_5 = sum(a)
s_5 =
    0.9475
%
echo off; diary off
```

s611x16

Estimate the sum of the series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}$ to 4 decimal places; i.e., to within an error of $\epsilon = 5 \times 10^{-5}$. Illustrate by graphing the terms of the sequence as well as the partial sums of the series.

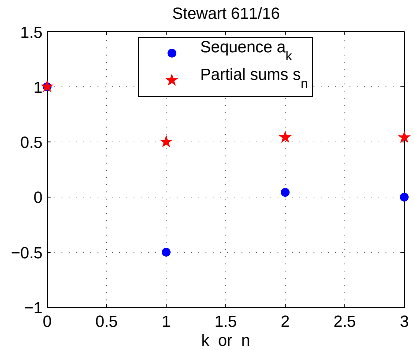
Solution

The ASET says that the magnitude of the error does not exceed that of the first neglected term of the series.

$$|R_N| \leq \frac{1}{(2(N+1))!} \overset{\text{want}}{\leq} \epsilon = 5 \times 10^{-5}$$

From the table below, choose $N = 3$ terms. In this case we have $s_4 \approx 0.5403$. (Yes, the 4th partial sum: the lower index of summation is 0, not 1.) Notice how rapidly the series converges.

N	0	1	2	3	4
$\dfrac{1}{(2\,(N+1)\,)!}$	0.5	0.417	0.0014	2×10^{-5}	3×10^{-7}




```
%
% Stewart 611/16
%
% Sequence a
%
n = 0:3; a = (-1).^n ./ factorial(2.*n);
plot(n,a,'bo', 'MarkerFaceColor', 'b', ...
    'MarkerSize', 5)
xlabel('k  or  n')
title('Stewart 611/16')
grid on; hold on
%
% Partial sums s
%
s = cumsum(a);
plot(n,s,'rp', 'MarkerFaceColor', 'r', ...
    'MarkerSize', 7)
legend('Sequence a_k', 'Partial sums s_n', ...
    'Location', 'North')
axis([0 3 -1 1.5])
%
% Approximate sum of infinite series: s_4
%
s_4 = s(4)
s_4 =
    0.5403
%
echo off; diary off
```