Spring 2005 Math 152
10 Infinite Sequences and Series
10.5 Power Series
Mon, 04/Apr ©2005, Art Belmonte

Summary

Definition

A **power series** centered at $x = a$ (or about a) has the form

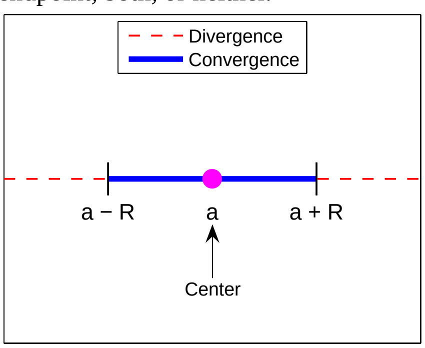
$$\sum_{n=0}^{\infty} c_n (x-a)^n$$

where the c_n are constant **coefficients** and x is a variable. The series may converge for some values of x, yet diverge for others.

For a particular power series, there are exactly three possibilities. The first two of these are common, whereas the third case is very unusual (and not interesting).

THEOREM

1. There is a positive real number R, called the **radius of convergence**, such that the series *converges* for $|x-a| < R$ and *diverges* for $|x-a| > R$. The **interval of convergence** I is finite and has one of four forms, depending on whether the series converges at the interval's left endpoint, right endpoint, both, or neither.



2. The series *converges* for all real x. Its radius of convergence is $R = \infty$ and its interval of convergence the entire real line.

3. The series converges (to 0) *only* at $x = a$. Its radius of convergence is $R = 0$ and its (degenerate) interval of convergence is $I = \{a\}$, a single point.

Hand Examples

617/6

Find the radius of convergence R and interval of convergence I for the power series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{\sqrt[3]{n}}$.

Solution

- For the series to converge by the Ratio Test, we need
$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{\sqrt[3]{n+1}}\cdot\frac{\sqrt[3]{n}}{x^n}\right| = \lim_{n\to\infty}\frac{|x|}{\sqrt[3]{1+\frac{1}{n}}} = |x| < 1.$$
Therefore, $R = 1$ and the center of I is $x = 0$.
- At $x = -1$, the left endpoint of I, the series is $\sum \frac{1}{n^{1/3}}$, a divergent p-series ($p = \frac{1}{3} \le 1$).
- At $x = 1$, the right endpoint of I, the series is $\sum \frac{(-1)^n}{n^{1/3}}$, which converges by the AST since $\frac{1}{n^{1/3}} \downarrow 0$.
- Therefore, $I = (-1, 1]$.

617/10

Find the radius of convergence R and interval of convergence I for the power series $\displaystyle\sum_{n=0}^{\infty} \frac{n^2 x^n}{10^n}$.

Solution

- For the series to converge, the Root Test (together with the GFF) requires
$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\frac{\sqrt[n]{n^2}\,|x|}{10} = \frac{|x|}{10} < 1 \text{ or } |x| < 10.$$
Therefore, $R = 10$ and the center of I is $x = 0$.
- At $x = -10$, I's left endpoint, the series is $\sum (-1)^n n^2$, which diverges by the Test for Divergence due to the fact that $\lim_{n\to\infty} (-1)^n n^2 \neq 0$.
- At $x = 10$, the right endpoint of I, the series is $\sum n^2$, which diverges by the Test for Divergence since $\lim_{n\to\infty} n^2 \neq 0$.
- Therefore, $I = (-10, 10)$.

617/12

Find the radius of convergence R and interval of convergence I for the power series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-1}}{(2n-1)!}$.

Solution

- For the series to converge by the Ratio Test, we need
$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{x^{2(n+1)-1}}{(2(n+1)-1)!}\cdot\frac{(2n-1)!}{x^{2n-1}}\right| \\ &= \lim_{n\to\infty}\frac{|x|^2}{(2n)(2n+1)} = 0 < 1.\end{aligned}$$
Since this is true for *all* x, we have $R = \infty$ and $I = \mathbb{R}$, the entire real line.

Example A

Find the radius of convergence R and interval of convergence I for the power series $\displaystyle\sum_{n=1}^{\infty} n^n x^n$.

Solution

For the series to converge by the Root Test, we need

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} (n\,|x|) < 1.$$

This holds *only* for $x = 0$ (in which case the limit is 0). Therefore, $R = 0$ and $I = \{0\}$.

617/14

Find the radius of convergence R and interval of convergence I for the power series $\displaystyle\sum_{n=1}^{\infty} \frac{(x-4)^n}{(5^n)\,n}$.

Solution

- The series converges via the Root Test (w/ GFF) provided

  $$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{|x-4|}{5\sqrt[n]{n}} = \frac{|x-4|}{5} < 1 \text{ or } |x-4| < 5.$$

  Therefore, $R = 5$ and the center of I is $x = 4$.

- At $x = -1$, the left endpoint of I, the series is $\sum \frac{(-1)^n}{n}$, which converges by the AST since $\frac{1}{n} \downarrow 0$.

- At $x = 9$, the right endpoint of I, the series is $\sum \frac{1}{n}$, a p-series ($p = 1 \le 1$) which diverges.

- Therefore, $I = [-1, 9)$.

Example B

Find the radius of convergence R and interval of convergence I for the power series $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^n\,(x-1)^n}{\sqrt{n+7}}$.

Solution

- For the series to converge by the Root Test (w/ GFF), we need

  $$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{3\,|x-1|}{\left(\sqrt[n]{n+7}\right)^{1/2}} = \frac{3\,|x-1|}{1} < 1 \text{ or } |x-1| < \frac{1}{3}.$$

  Therefore, $R = \frac{1}{3}$ and the center of I is $x = 1$.

- At $x = \frac{2}{3}$, I's left endpoint, the series is $\sum \frac{1}{\sqrt{n+7}}$, which is asymptotically similar to the divergent p-series $\sum \frac{1}{\sqrt{n}}$ ($p = \frac{1}{2} \le 1$). Hence $\sum \frac{1}{\sqrt{n+7}}$ diverges by the Limit Comparison Test since $\dfrac{a_n}{b_n} = \dfrac{\sqrt{n}}{\sqrt{n+7}} = \sqrt{\dfrac{1}{1+\frac{7}{n}}} \to 1 > 0.$

- At $x = \frac{4}{3}$, the right endpoint of I, the series is $\sum \frac{(-1)^n}{\sqrt{n+7}}$, which converges by the AST since $\frac{1}{\sqrt{n+7}} \downarrow 0$.

- Therefore, $I = \left(\frac{2}{3}, \frac{4}{3}\right]$.

MATLAB Examples

s617x21

The function J_1 defined by

$$J_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n\,x^{2n+1}}{n!\,(n+1)!2^{2n+1}}$$

is called the *Bessel function of order* 1.

- Find its domain.

- Graph the first several partial sums on the same figure.

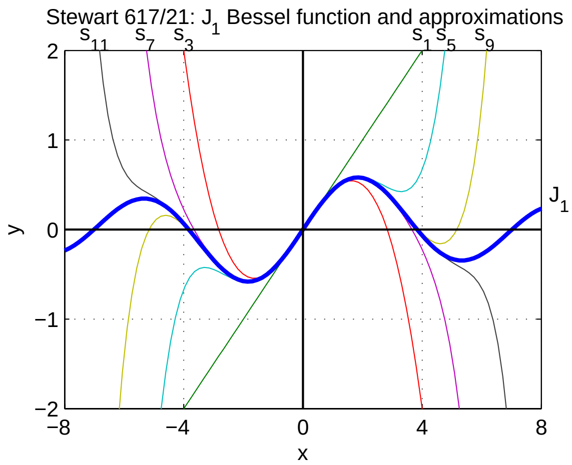
- If your CAS has built-in Bessel functions, graph J_1 on the same figure as the partial sums in part (b) and observe how the partial sums approximate J_1.

Solution

For the series to converge by the Ratio Test, we need

$$\begin{aligned}
&\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \\
= &\lim_{n\to\infty} \left|\frac{x^{2(n+1)+1}}{(n+1)!\,((n+1)+1)!2^{2(n+1)+1}} \cdot \frac{n!\,(n+1)!2^{2n+1}}{x^{2n+1}}\right| \\
= &\lim_{n\to\infty} \left|\frac{x^{2n+3}}{(n+1)!\,(n+2)!2^{2n+3}} \cdot \frac{n!\,(n+1)!2^{2n+1}}{x^{2n+1}}\right| \\
= &\lim_{n\to\infty} \frac{|x|^2}{4\,(n+1)\,(n+2)} = 0 < 1.
\end{aligned}$$

This is true for *all* x. The domain of J_1 is $\mathbb{R}$, the entire real line.




```
%
% Stewart 617/21: J_1 and some of its partial sums
% (or, "The party's gone out of bounds..." - the B-52s)
%
% Partial sums of J_1, Bessel function of order 1
% (This segment of code is obscure; don't sweat it!)
format rat
c = zeros(1,12); T = zeros(6,12);
n = 0:5;
c(2*n+2) = (-1).^n ./ ...
    ( factorial(n) .* factorial(n+1) .* 2.^(2.*n+1) );
c = fliplr(c);
c'; % an interim check...
for k = n
    T(k+1,:) = c;
    T(k+1, 1:10-2*k) = 0;
end
format short
%
M = zeros(6,100);
x = linspace(-8, 8);
J1 = besselj(1, x);
for j = n+1
    M(j,:) = polyval(T(j,:),x);
end
plot(x,J1, x,M(1,:), x,M(2,:), x,M(3,:), ...
    x,M(4,:), x,M(5,:), x,M(6,:))
grid on; hold on
% (Used initially to identify the players)
% legend('J_1', 's_1', 's_3', 's_5', ...
%     's_7', 's_9', 's_{11}', 'Location', 'South')
plot(x,J1, 'LineWidth', 2)
plot([-8 8], [0 0], 'k', 'LineWidth', 1)
plot([0 0], [-2 2], 'k', 'LineWidth', 1)
axis([-8 8 -2 2])
set(gca, 'Xtick', -8 : 4: 8)
set(gca, 'Ytick', -2:2)
xlabel('x'); ylabel('y')
title('Stewart 617/21: J_1 Bessel function and approximations')
%
text(8.6, 0.3, 'J_1', 'HorizontalAlignment', 'Center')
text(4.0, 2.1, 's_1', 'HorizontalAlignment', 'Center')
text(4.8, 2.1, 's_5', 'HorizontalAlignment', 'Center')
text(6.1, 2.1, 's_9', 'HorizontalAlignment', 'Center')
text(-4.0, 2.1, 's_3', 'HorizontalAlignment', 'Center')
text(-5.3, 2.1, 's_7', 'HorizontalAlignment', 'Center')
text(-7.0, 2.1, 's_{11}', 'HorizontalAlignment', 'Center')
%
echo off; diary off
```