Spring 2005 Math 152
10 Infinite Sequences and Series
10.6 Representations of Functions as Power Series
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Summary

THEOREM

A power series $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ having a radius of convergence $R > 0$ is differentiable and integrable on the interior of its interval of convergence; i.e., $|x-a| < R$ or $(a-R, a+R)$. Essentially, we may differentiate or integrate term-by-term.

$$f'(x) = \frac{d}{dx}\left(\sum_{n=0}^{\infty} c_n (x-a)^n\right) = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$$

$$\int f(x)\,dx = \int \sum_{n=0}^{\infty} c_n (x-a)^n \,dx = C + \sum_{n=0}^{\infty} \frac{c_n (x-a)^{n+1}}{n+1}$$

The radii of convergence of the derivative and antiderivative series representations are each equal to R. (NOTE: The *intervals* of convergence of these representations may not be the same be the same as the *interval* of convergence I of the original power series. At the endpoints of I, if any, convergence must still be checked.)

A way to obtain the series representation of a function

Starting with the geometric series $\sum_{n=0}^{\infty} x^n$, which converges to $\frac{1}{1-x}$ for $|x| < 1$ and diverges otherwise, we *may* be able to obtain series representations for other functions by algebraic manipulation, differentiation or indefinite integration, and/or use of the preceding theorem.

While this is by no means systematic, it will have to do until a more sure-fire technique is introduced in the next section: Taylor and Maclaurin series.

Hand Examples

Example A

Find a power series representation for $f(x) = \frac{2}{3x+4}$ and determine its radius and interval of convergence.

Solution

- Use algebraic manipulation together with the Geometric Series Theorem (GST).

$$\begin{aligned}
\frac{2}{3x+4} &= \frac{2}{4\left(1-\left(-\frac{3}{4}x\right)\right)} \\
&= \frac{1}{2}\cdot\frac{1}{1-\left(-\frac{3}{4}x\right)} \\
&= \frac{1}{2}\cdot\sum_{n=0}^{\infty}\left(-\frac{3}{4}x\right)^n, \quad \text{for } \left|-\frac{3}{4}x\right| < 1 \text{ via GST} \\
&= \sum_{n=0}^{\infty}\frac{(-3)^n x^n}{2^{2n+1}}
\end{aligned}$$

  We require $\left|-\frac{3}{4}x\right| < 1$ or $|x| < \frac{4}{3}$. Therefore, $R = \frac{4}{3}$ and the center of I is $x = 0$.

- At $x = -\frac{4}{3}$, the left endpoint of I, the series is $\sum \frac{1}{2}$, which diverges by the Test for Divergence since $\lim \frac{1}{2} \neq 0$.
- At $x = \frac{4}{3}$, the right endpoint of I, the series is $\sum\left(-\frac{1}{2}\right)$, which diverges by oscillation.
- Therefore, $I = \left(-\frac{4}{3}, \frac{4}{3}\right)$.

Example B

Find a power series representation for $f(x) = \frac{x}{x^2-3x+2}$ and determine its radius and interval of convergence.

Solution

- First split the expression into a sum of partial fractions.

$$\begin{aligned}
\frac{x}{(x-1)(x-2)} &= \frac{A}{x-1} + \frac{B}{x-2} \\
x &= A(x-2) + B(x-1) \\
x + 0 &= (A+B)x + (-2A-B)
\end{aligned}$$

  Thus $A + B = 1$ and $-2A + -B = 0$. So $B = -2A$ whence $-A = 1$ or $A = -1$ and $B = 2$. Therefore,

$$\frac{x}{x^2-3x+2} = \frac{-1}{x-1} + \frac{2}{x-2}.$$

- Now use algebraic manipulation and the GST.

$$\begin{aligned}
\frac{1}{1-x} - \frac{2}{2-x} &= \frac{1}{1-x} - \frac{1}{1-\frac{1}{2}x} \\
\text{If } |x| < 1 \text{ and } \left|\frac{1}{2}x\right| < 1 \text{ via GST:} \quad &= \sum_{n=0}^{\infty} x^n - \sum_{n=0}^{\infty}\left(\frac{1}{2}x\right)^n \\
&= \sum_{n=0}^{\infty}\left(1-\frac{1}{2^n}\right)x^n \\
&= \sum_{n=1}^{\infty}\left(1-\frac{1}{2^n}\right)x^n
\end{aligned}$$

We require *both* $|x| < 1$ *and* $\left|\frac{1}{2}x\right| < 1$; that is, both $|x| < 1$ and $|x| < 2$. In other words, we need $|x| < 1$. Therefore, $R = 1$ and the center of I is $x = 0$.

- At $x = -1$, the left endpoint of I, the series is $\sum a_n = \sum \left(1 - \frac{1}{2^n}\right)(-1)^n$, which diverges by the Test for Divergence since $\lim a_n \neq 0$.
- At $x = 1$, the right endpoint of I, the series is $\sum a_n = \sum \left(1 - \frac{1}{2^n}\right)$, which diverges by the Test for Divergence since $\lim a_n \neq 0$.
- Therefore, $I = (-1, 1)$.

622/6

Find a power series representation for $f(x) = \dfrac{1}{x^4 + 16}$ and determine its radius and interval of convergence.

Solution

- Use algebraic manipulation and the GST.

$$\begin{aligned}
\frac{1}{x^4+16} &= \frac{1}{16\left(1-\left(-\frac{1}{16}x^4\right)\right)} \\
&= \frac{1}{16}\cdot\frac{1}{1-\left(-\frac{1}{16}x^4\right)} \\
&= \frac{1}{16}\cdot\sum_{n=0}^{\infty}\left(-\frac{1}{16}x^4\right)^n, \quad \text{if } \left|-\frac{1}{16}x^4\right| < 1 \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n x^{4n}}{16^{n+1}}
\end{aligned}$$

We require $\left|-\frac{1}{16}x^4\right| < 1$ or $|x|^4 < 16$ or $|x| < 2$. Therefore, $R = 2$ and the center of I is $x = 0$.

- At $x = -2$, the left endpoint of I, the series is $\sum \frac{(-1)^n}{16}$, which diverges by oscillation.
- At $x = 2$, the right endpoint of I, the series is also $\sum \frac{(-1)^n}{16}$, which diverges by oscillation.
- Therefore, $I = (-2, 2)$.

622/10

Find a power series representation for $f(x) = \ln(1 + x)$ and determine its radius and interval of convergence.

Solution

- Note that $\ln(1 + x)$ is an antiderivative of $\dfrac{1}{1+x}$.

$$\begin{aligned}
\ln(1+x) &= \int \frac{1}{1+x}\,dx = \int \frac{1}{1-(-x)}\,dx \\
&= \int \sum_{n=0}^{\infty}(-x)^n\,dx, \quad \text{if } |-x| < 1 \\
&= C + \sum_{n=0}^{\infty}\left((-1)^n \int x^n\,dx\right) \\
&= C + \sum_{n=0}^{\infty}(-1)^n\,\frac{x^{n+1}}{n+1} \\
\ln(1+x) &= C + \sum_{k=1}^{\infty}\frac{(-1)^{k-1}x^k}{k} \\
0 = \ln(1+0) &= C \\
\text{Thus } \ln(1+x) &= \sum_{k=1}^{\infty}\frac{(-1)^{k-1}x^k}{k}.
\end{aligned}$$

We require $|-x| < 1$ or $|x| < 1$. Therefore, $R = 1$ and the center of I is $x = 0$.

- At $x = -1$, the left endpoint of I, the series is $-\sum \frac{1}{k}$, which diverges to $-\infty$ since the p-series $\sum \frac{1}{k}$ diverges to ∞ $(p = 1 \leq 1)$.
- At $x = 1$, the right endpoint of I, the series is $\sum \frac{(-1)^{k-1}}{k}$, which converges by the AST since $\frac{1}{k} \downarrow 0$.
- Therefore, $I = (-1, 1]$.

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Find a power series representation for $f(x) = \dfrac{1}{(1+x)^3}$ and determine its radius and interval of convergence.

Solution

- By 622/10 we have $\ln(1+x) = \sum_{k=1}^{\infty}\frac{(-1)^{k-1}x^k}{k}$ for $x \in (-1, 1]$. Hence

$$\begin{aligned}
(1+x)^{-1} &= \frac{1}{1+x} \\
&= \frac{d}{dx}(\ln(1+x)) \\
&= \frac{d}{dx}\left(\sum_{k=1}^{\infty}\frac{(-1)^{k-1}x^k}{k}\right) \\
&= \sum_{k=1}^{\infty}(-1)^{k-1}x^{k-1} \\
&= \sum_{n=0}^{\infty}(-1)^n x^n,
\end{aligned}$$

which holds for $x \in I = (-1, 1)$ but clearly not at the endpoints of I. Thus $(1+x)^{-1} = \sum_{n=0}^{\infty}(-1)^n x^n$ for $|x| < 1$.

- Repeat this derivative trick.

$$\begin{aligned}(1+x)^{-2} &= \frac{d}{dx}\left(-(1+x)^{-1}\right)\\ &= \frac{d}{dx}\left(-\sum_{n=0}^{\infty}(-1)^n x^n\right)\\ &= \sum_{n=1}^{\infty}(-1)^{n+1} n x^{n-1}\\ &= \sum_{n=1}^{\infty}(-1)^{n-1} n x^{n-1}\\ &= \sum_{k=0}^{\infty}(-1)^k (k+1)\, x^k, \quad \text{for } |x|<1.\end{aligned}$$

- Hey campers, the third time's the charm!

$$\begin{aligned}(1+x)^{-3} &= \frac{d}{dx}\left(-\tfrac{1}{2}(1+x)^{-2}\right)\\ &= \frac{d}{dx}\left(-\frac{1}{2}\sum_{k=0}^{\infty}(-1)^k (k+1)\, x^k\right)\\ &= -\frac{1}{2}\sum_{k=1}^{\infty}(-1)^k (k+1)\, k x^{k-1}\\ &= \sum_{k=1}^{\infty}\frac{(-1)^{k+1}(k+1)\, k x^{k-1}}{2}\\ &= \sum_{n=0}^{\infty}\frac{(-1)^{n+2}(n+2)(n+1)\, x^n}{2}\\ &= \sum_{n=0}^{\infty}\frac{(-1)^{n}(n+2)(n+1)\, x^n}{2}, \quad \text{for } |x|<1.\end{aligned}$$

- We conclude that
$$\frac{1}{(1+x)^3} = \sum_{n=0}^{\infty}\frac{(-1)^n (n+2)(n+1)\, x^n}{2}, \quad \text{for } |x|<1.$$

622/12

Find a power series representation for $f(x) = x\ln(1+x)$ and determine its radius and interval of convergence.

Solution

- By 622/10 we have
$$x\ln(1+x) = x\sum_{k=1}^{\infty}\frac{(-1)^{k-1}x^k}{k} = \sum_{k=1}^{\infty}\frac{(-1)^{k-1}x^{k+1}}{k}.$$

- For the series to converge by the Ratio Test, we need
$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{x^{k+2}}{k+1}\cdot\frac{k}{x^{k+1}}\right| = \lim_{k\to\infty}\frac{|x|}{1+\frac{1}{k}} = |x| < 1.$$
Therefore, $R=1$ and the center of I is $x=0$.

- At $x=-1$, the left endpoint of I, the series is $\sum \frac{1}{k}$, the divergent harmonic series.

- At $x=1$, the right endpoint of I, the series is $\sum \frac{(-1)^{k-1}}{k}$, which converges by the AST since $\frac{1}{k}\downarrow 0$.

- Therefore, $I=(-1,1]$.

622/22

Evaluate the indefinite integral $\int \tan^{-1}\left(x^2\right)\,dx$ as a power series.

Solution

- From Example 7 on page 621 of Stewart we have
$$\tan^{-1} w = \sum_{n=0}^{\infty}(-1)^n\,\frac{w^{2n+1}}{2n+1}, \quad \text{for } |w|\le 1.$$

- Thus $\tan^{-1}\left(x^2\right) = \sum_{n=0}^{\infty}(-1)^n\,\frac{x^{4n+2}}{2n+1}, \quad \text{for } |x|\le 1.$

- Hence
$$\int \tan^{-1}\left(x^2\right)\,dx = C + \sum_{n=0}^{\infty}\frac{(-1)^n x^{4n+3}}{(2n+1)(4n+3)}, \quad \text{for } |x|<1.$$

622/28

Show that the function
$$f(x) = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$$
is a solution of the differential equation
$$f''(x) + f(x) = 0.$$

Solution

- We have
$$f'(x) = \sum_{n=1}^{\infty}\frac{(-1)^n (2n)\, x^{2n-1}}{(2n)!} = \sum_{n=1}^{\infty}\frac{(-1)^n x^{2n-1}}{(2n-1)!}.$$

- In turn,
$$\begin{aligned}f''(x) &= \sum_{n=1}^{\infty}\frac{(-1)^n (2n-1)\, x^{2n-2}}{(2n-1)!}\\ &= \sum_{n=1}^{\infty}\frac{(-1)^n x^{2n-2}}{(2n-2)!}\\ &= \sum_{n=1}^{\infty}\frac{(-1)^{(n-1)+1} x^{2(n-1)}}{\left(2(n-1)\right)!}\\ &= \sum_{k=0}^{\infty}\frac{(-1)^{k+1} x^{2k}}{(2k)!}\\ &= -\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}.\end{aligned}$$

- Hence
$$f''(x)+f(x) = \left(-\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}\right)+\left(\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}\right) = 0.$$

- Therefore, $f(x) = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$ is a solution of the stated differential equation.

MATLAB Examples

s622x16

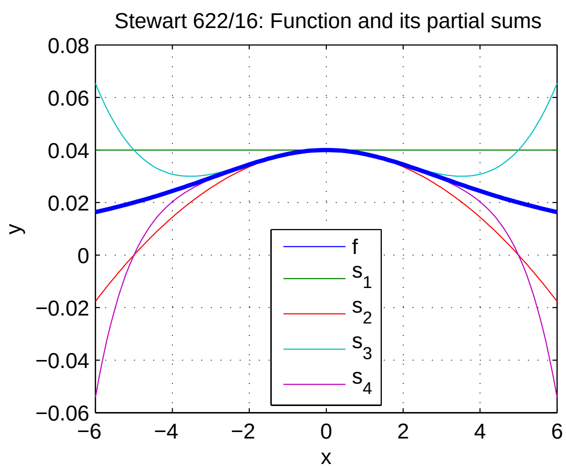
Find a power series representation for $f(x) = \dfrac{1}{x^2 + 25}$ and graph $f(x)$ and several partial sums $s_n(x)$ on the same plot.

Solution

Use algebraic manipulation and the GST.

$$\begin{aligned}
\frac{1}{x^2+25} &= \frac{1}{25\left(1-\left(-\frac{1}{25}x^2\right)\right)} \\
&= \frac{1}{25}\cdot\frac{1}{1-\left(-\frac{1}{25}x^2\right)} \\
&= \frac{1}{25}\cdot\sum_{n=0}^{\infty}\left(-\frac{1}{25}x^2\right)^n, \quad \text{if } \left|-\frac{1}{25}x^2\right| < 1 \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{25^{n+1}}
\end{aligned}$$

We require $\left|-\frac{1}{25}x^2\right| < 1$ or $|x|^2 < 25$ or $|x| < 5$. Therefore, $R = 5$ and the center of I is $x = 0$. Each successive partial sum of the series representation approximates the graph of $f(x)$ more closely.




```
%
% Stewart 622/16: Graph of f and some of the
% partial sums from its series representation
%
x = linspace(-6, 6, 121);
f = 1 ./ (x.^2 + 25);
s_1 = 0.*x + 1./25;
s_2 = 1./25 - x.^2./625;
s_3 = 1./25 - x.^2./625 + x.^4./15625;
s_4 = 1./25 - x.^2./625 + x.^4./15625 - x.^6./390625;
plot(x,f, x,s_1, x,s_2, x,s_3, x,s_4)
legend('f', 's_1', 's_2', 's_3', 's_4', ...
    'Location', 'South')
xlabel('x'); ylabel('y')
title('Stewart 622/16: Function and its partial sums')
grid on; hold on
plot(x, f, 'LineWidth', 2)
%
echo off; diary off
```

622/24

Use a power series to approximate the definite integral

$$\int_0^{1/2} \tan^{-1}\left(x^2\right)\,dx$$

to six decimal places ($\epsilon = 5 \times 10^{-7}$).

Solution

- Using the alternating series from 622/22, we have

$$\begin{aligned}
\int_0^{1/2}\tan^{-1}\left(x^2\right)dx &= \left(\sum_{n=0}^{\infty}\frac{(-1)^n x^{4n+3}}{(2n+1)(4n+3)}\right)\Bigg|_{x=0}^{x=1/2} \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n\left(\frac{1}{2}\right)^{4n+3}}{(2n+1)(4n+3)}.
\end{aligned}$$

- By the ASET, we need

$$|R_N| \le |a_{N+1}| = \frac{1}{(2N+3)(4N+7)\,2^{4n+7}} \le 5\times 10^{-7}$$

which implies $N \ge 1.84$. So choose $N = 2$, for which the partial sum of the series is

$$\sum_{n=0}^{2}\frac{(-1)^n\left(\frac{1}{2}\right)^{4n+3}}{(2n+1)(4n+3)} \approx 0.0413.$$

```
%
% Stewart 622/24
%
syms N
v = 1 / ( (2*N+3) * (4*N+7) * 2^(4*N+7) ); pretty(v)

                                  1
                  ------------------------------
                                           (4 N + 7)
                  (2 N + 3) (4 N + 7) 2
N = 0:3;
v = eval(vectorize(v));
table = [N' v']';
fprintf('%2i %9.2e\n', table)
 0  3.72e-04
 1  8.88e-06
 2  2.91e-07
 3  1.12e-08
%
syms n
a = (-1)^n / ( (2*n+1) * (4*n+3) * 2^(4*n+3) ); pretty(a)

                                 n
                             (-1)
                  ------------------------------
                                           (4 n + 3)
                  (2 n + 1) (4 n + 3) 2
n = 0:2; a = eval(vectorize(a));
s_2 = sum(a)
s_2 =
    0.0413
%
echo off; diary off
```