Spring 2005 Math 152
10 Infinite Sequences and Series
10.9 Taylor Polynomial Applications
Wed, 13/Apr ©2005, Art Belmonte

Summary

In this section, we apply the theory and formulas developed in Section 10.7 to problems of a graphical nature. In this regard, we'll dispense with hand work (mostly) and resort to unabashed use of machine firepower. Recall these pertinent concepts.

- **Taylor series**:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)(x-a)^n}{n!} \quad \text{for } |x-a| < R.$$

- **Taylor polynomials**:

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)(x-a)^k}{k!}, \quad n = 0, 1, 2, \ldots$$

- **Taylor's formula for remainder** $R_n(x) = f(x) - T_n(x)$:

$$R_n(x) = \frac{f^{(n+1)}(z)(x-a)^{n+1}}{(n+1)!},$$

  where z is strictly between x and a.

- **Taylor's Inequality**: If $\left|f^{(n+1)}(x)\right| \le M$ for $|x-a| < R$, then the remainder $R_n(x)$ of the Taylor series satisfies

$$|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1} \quad \text{for } |x-a| < R.$$

MATLAB Examples

MATLAB can compute Taylor polynomials with one command! Given a positive integer n, **taylor(f,n)** returns the $(n-1)^{\text{st}}$ Taylor polynomial of f about 0, whereas **taylor(f,n,a)** centers the Taylor polynomial at a. (The TI-89 also has a **taylor** command.)

s644x02

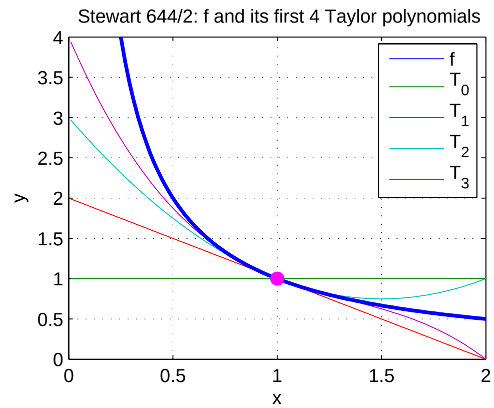
(a) Find the Taylor polynomials up to degree 3 for $f(x) = 1/x$ centered at $a = 1$. Graph f and these polynomials on a common plot.

(b) Evaluate f and these polynomials at $x = 0.9$ and 1.3.

(c) Comment on how the Taylor polynomials converge to $f(x)$.

Solution

(a) We have $T_n(x) = \sum_{k=0}^{n} (-1)^k (x-1)^k$, $n = 0, 1, 2, 3$.



(b) Here is a table of values from the diary file.

x	f	T_0	T_1	T_2	T_3
0.9	1.1111	1.0000	1.1000	1.1100	1.1110
1.3	0.7692	1.0000	0.7000	0.7900	0.7630

(c) The higher the degree of the Taylor polynomial, the more accurately its graph matches the graph of f.

```
%
% Stewart 644/2
%
syms x
f = 1 / x; pretty(f) % function

                                     1/x
%
T = sym(zeros(4,1)); a = 1;
for k = 1:4
    T(k) = taylor(f,k,a);
    T(k) = taylor(f,k,a);
    T(k) = taylor(f,k,a);
    T(k) = taylor(f,k,a);
end
% First 4 Taylor polynomials
pretty(T)

                          [             1              ]
                          [                            ]
                          [           2 - x            ]
                          [                            ]
                          [                     2      ]
                          [     2 - x + (x - 1)        ]
                          [                            ]
                          [                2          3]
                          [2 - x + (x - 1)   - (x - 1) ]
% (b) Values of f and polys at selected values
x = [0.9 1.3];
ff = eval(vectorize(f));
T0 = 0.*x + 1;
T1 = eval(vectorize(T(2)));
T2 = eval(vectorize(T(3)));
T3 = eval(vectorize(T(4)));
table = [x' ff' T0' T1' T2' T3']
table =
    0.9000    1.1111    1.0000    1.1000    1.1100    1.1110
    1.3000    0.7692    1.0000    0.7000    0.7900    0.7630
% (c) Plot
x = linspace(0.01, 2);
ff = eval(vectorize(f));
T0 = 0.*x + 1;
```

```
T1 = eval(vectorize(T(2)));
T2 = eval(vectorize(T(3)));
T3 = eval(vectorize(T(4)));
plot(x,ff, x,T0, x,T1, x,T2, x,T3)
grid on; axis([0 2 0 4]); hold on
legend('f', 'T_0', 'T_1', 'T_2', 'T_3', ...
    'Location', 'NorthEast')
plot(x,ff, 'LineWidth', 2)
plot(1, 1, 'mo', 'MarkerFaceColor', 'm', ...
    'MarkerSize', 7)
xlabel('x'); ylabel('y')
title('Stewart 644/2: f and its first 4 Taylor polynomials')
%
echo off; diary off
```

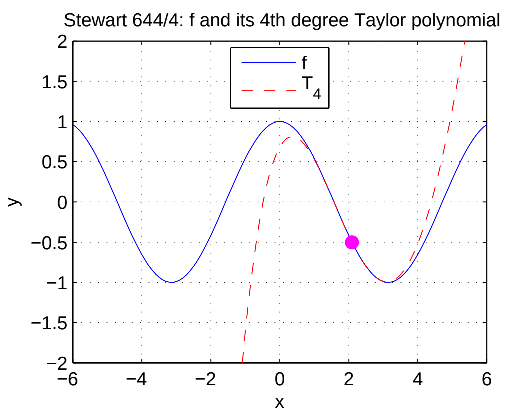
s644x04

Find $T_4(x)$, the fourth degree Taylor polynomial of $f(x) = \cos x$ centered at $a = \frac{2}{3}\pi$. Graph f and T_4 on the same plot.

Solution

We have

$$\begin{aligned} T_4(x) &= -\tfrac{1}{2} - \tfrac{1}{2}\sqrt{3}\left(x - \tfrac{2}{3}\pi\right) + \tfrac{1}{4}\left(x - \tfrac{2}{3}\pi\right)^2 \\ &\quad + \tfrac{1}{12}\sqrt{3}\left(x - \tfrac{2}{3}\pi\right)^3 - \tfrac{1}{48}\left(x - \tfrac{2}{3}\pi\right)^4. \end{aligned}$$



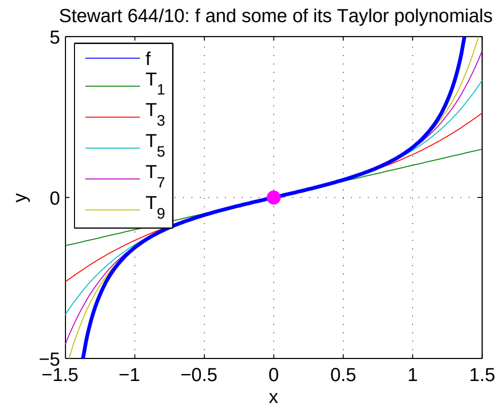

```
%
% Stewart 644/4
%
syms x
f = cos(x); % function
T4 = taylor(f, 5, 2*pi/3); % Taylor polynomial of degree 4
pretty(T4)

                  1/2                                 2
  - 1/2 - 1/2 3     (x - 2/3 pi) + 1/4 (x - 2/3 pi)
             1/2                3                      4
  + 1/12 3     (x - 2/3 pi)       - 1/48 (x - 2/3 pi)
% Plot
x = linspace(-6, 6, 121);
f = eval(vectorize(f));
T4 = eval(vectorize(T4));
plot(x,f, x,T4, 'r--')
grid on; axis([-6 6 -2 2]); hold on
legend('f', 'T_4', 'Location', 'North')
plot(2*pi/3, cos(2*pi/3), 'mo', 'MarkerFaceColor', 'm', ...
    'MarkerSize', 7)
xlabel('x'); ylabel('y')
title('Stewart 644/4: f & its 4th degree Taylor polynomial')
%
echo off; diary off
```

s644x10

Find the Taylor polynomials T_1, T_3, T_5, T_7, T_9 at $a = 0$ for $f(x) = \tan x$. Graph these polynomials and f on the same screen.

Solution

You know the drill by now. . .




```
%
% Stewart 644/10
%
syms x
f = tan(x); % function
%
T = sym(zeros(5,1)); a = 1;
for k = 1:5
    T(k) = taylor(f,2*k);
    T(k) = taylor(f,2*k);
    T(k) = taylor(f,2*k);
    T(k) = taylor(f,2*k);
    T(k) = taylor(f,2*k);
end
% Taylor polynomials of degree 1, 3, 5, 7, 9
pretty(T)

                   [                    x                      ]
                   [                                           ]
                   [                            3              ]
                   [               x + 1/3 x                   ]
                   [                                           ]
                   [                     3         5           ]
                   [          x + 1/3 x  + 2/15 x              ]
                   [                                           ]
                   [                3         5    17    7     ]
                   [      x + 1/3 x  + 2/15 x  + --- x         ]
                   [                             315           ]
                   [                                           ]
                   [          3         5    17    7    62    9]
                   [x + 1/3 x  + 2/15 x  + --- x  + ---- x     ]
                   [                       315      2835       ]
% Plot
x = linspace(-1.5, 1.5, 301);
f = eval(vectorize(f));
T1 = eval(vectorize(T(1)));
T3 = eval(vectorize(T(2)));
T5 = eval(vectorize(T(3)));
T7 = eval(vectorize(T(4)));
T9 = eval(vectorize(T(5)));
plot(x,f, x,T1, x,T3, x,T5, x,T7, x,T9)
```

```
grid on; axis([-1.5 1.5 -5 5]); hold on
legend('f', 'T_1', 'T_3', 'T_5', 'T_7', 'T_9', ...
    'Location', 'NorthWest')
plot(x,f, 'LineWidth', 2)
plot(0, 0, 'mo', 'MarkerFaceColor', 'm', ...
    'MarkerSize', 7)
xlabel('x'); ylabel('y')
title('Stewart 644/10: f & some of its Taylor polynomials')
%
echo off; diary off
```

s644x12

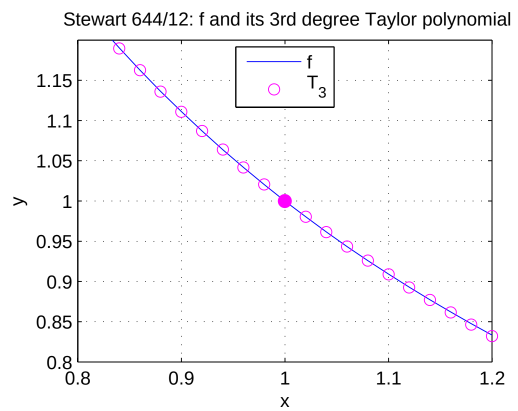
(a) Approximate $f(x) = 1/x$ at $a = 1$ by T_3 for $0.8 \le x \le 1.2$.

(b) Use Taylor's Inequality to estimate the accuracy of the approximation $f(x) \approx T_3(x)$ for $0.8 \le x \le 1.2$.

(c) Check your result in (b) by graphing the remainder $|R_n(x)|$.

Solution

(a) Here is a plot of f and T_3; nice fit!

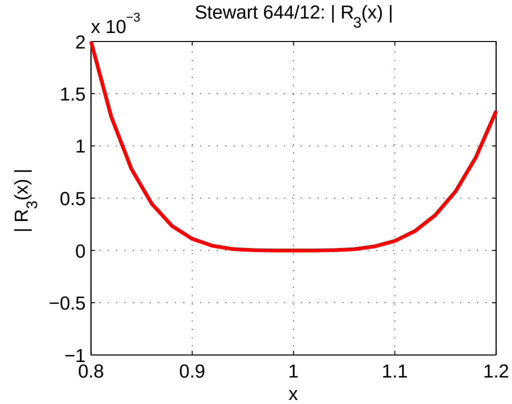


(b) Since $\left|f^{(3+1)}(x)\right| = \dfrac{24}{|x|^5} \le \dfrac{24}{|0.8|^5} = 73.2422 = M$ for $|x-1| < 0.2$, then the remainder $R_3(x)$ of the Taylor series satisfies

$$|R_3(x)| \le \frac{M}{(3+1)!}|x-1|^{3+1} \le \frac{73.2422}{24}(0.2)^4 = 4.88 \times 10^{-3}$$

for $|x-1| < 0.2$. Thus $U = 4.88 \times 10^{-3}$ is an *upper bound* on the absolute error.

(c) Here is a plot of $|R_3(x)|$, the actual magnitude of the error in the approximation of f by T_3. Note that $|R_3(x)| \le U$ throughout the interval $(0.8, 1.2)$ as guaranteed by the theory!




```
%
% Stewart 644/12
%
syms x
f = 1/x; % function
% (b)
D4f = diff(f,x,4); pretty(D4f)

                                              24
                                             ----
                                               5
                                              x
M = subs(D4f, x, 0.8)
M =
   73.2422
abs_max_err = M * 0.2^4 / factorial(4);
fprintf('Absolute maximum error: %6.2e\n', abs_max_err)
Absolute maximum error: 4.88e-03
T3 = taylor(f, 4, 1); % Taylor polynomial of degree 3
pretty(T3)

                                      2           3
                       2 - x + (x - 1)  - (x - 1)
% (a) Plot of f and T3
x = linspace(0.8, 1.2, 21);
fg = eval(vectorize(f));
T3g = eval(vectorize(T3));
plot(x,fg, x,T3g, 'mo')
grid on; axis([0.8, 1.2 0.8 1.2]); hold on
legend('f', 'T_3', 'Location', 'North')
plot(1, 1, 'mo', 'MarkerFaceColor', 'm', ...
    'MarkerSize', 7)
xlabel('x'); ylabel('y')
title('Stewart 644/12: f and its 3rd degree Taylor polynomial')
% (c) Plot of remainder R_3
R3 = fg - T3g;
figure
plot(x,abs(R3), 'r', 'LineWidth', 2); grid on
axis([0.8 1.2 -1e-3 2e-3])
xlabel('x'); ylabel('| R_3(x) | ')
title('Stewart 644/12: | R_3(x) |')
%
echo off; diary off
```

s644x22

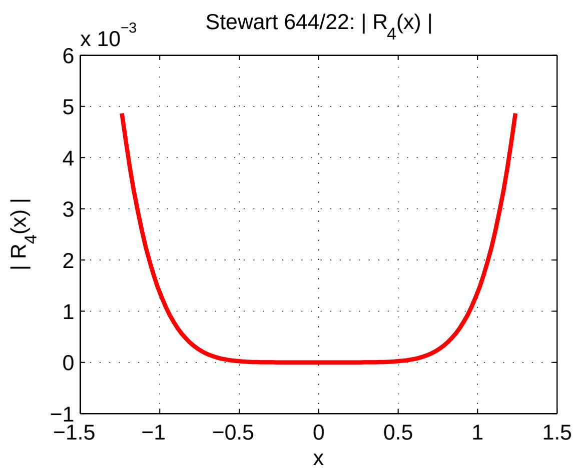
Use the ASET or Taylor's Inequality to estimate the range of values of x for which the approximation $T_4(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$ to $f(x) = \cos x$ is accurate to within $\epsilon = 0.005$. Check your answer graphically.

Solution

The Maclaurin series for $\cos x$ is $1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \cdots$. Via the ASET, the error $R_4(x)$ in the approximation $T_4(x)$ satisfies $|R_4(x)| \leq \frac{1}{720}x^6$, the magnitude of the first neglected term of the series. Let's see where this leads.

$$|R_4(x)| \leq \frac{x^6}{720} \overset{\text{want}}{\leq} 0.005 = \frac{1}{200}$$
$$|x| \leq \sqrt[6]{\frac{720}{200}} \approx 1.24$$

Therefore, the approximation $T_4(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$ is within $\epsilon = 0.005$ of $f(x)$ on the interval $(-1.24, 1.24)$. Here is a plot of $|R_4(x)|$ over this interval that verifies this. Beauty, eh?




```
%
% Stewart 644/22
%
syms x
f = cos(x); % function
T = taylor(f, 5); % Taylor polynomial of degree 4
c = (720/200)^(1/6)
c =
    1.2380
pretty(T)

                                            2           4
                                1 - 1/2 x   + 1/24 x
x = linspace(-c, c);
abs_R = eval(vectorize(abs(f-T)));
plot(x,abs_R, 'r', 'LineWidth', 2); grid on
axis([-1.5 1.5 -1e-3 6e-3])
xlabel('x'); ylabel('| R_4(x) |')
title('Stewart 644/22: | R_4(x) |')
%
echo off; diary off
```