Spring 2004 Math 253/501–503
11 Three Dimensional Analytic Geometry and Vectors
11.4 Equations of Lines and Planes
Thu, 22/Jan ©2004, Art Belmonte

Summary

- vector equation of a line: \( \mathbf{r} = \mathbf{L}(t) = \mathbf{a} + t\mathbf{v}, \, |t| < \infty \), where \( \mathbf{a} \) is the position vector of a point on the line, \( \mathbf{v} \) is a direction vector for the line, and \( t \) is a scalar.

- parametric equations of a line: \( \mathbf{r}(t) = [x(t), y(t), z(t)] = \mathbf{L}(t) = [a_1 + tv_1, a_2 + tv_2, a_3 + tv_3], \, |t| < \infty \).

- symmetric equations of a line: Solve the preceding for \( t \):
  \[
  \frac{x - a_1}{v_1} = \frac{y - a_2}{v_2} = \frac{z - a_3}{v_3}, \quad \text{provided} \ v_k \neq 0, \, k = 1, 2, 3.
  \]
  In the event that (say) \( v_1 = 0 \), then write \( x = a_1 \). Similarly, write \( y = a_2 \) if \( v_2 = 0 \) or \( z = a_3 \) if \( v_3 = 0 \).

- vector equation of a plane: \( \mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0 \) or \( \mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{r}_0 \), since a normal vector \( \mathbf{n} \) is perpendicular to any vector lying in the plane!

- scalar equation of a plane: Substituting \( \mathbf{n} = [n_1, n_2, n_3] \), \( \mathbf{r} = [x, y, z] \), and \( \mathbf{r}_0 = [x_0, y_0, z_0] \) into the first vector equation of a plane yields
  \[
  n_1 (x - x_0) + n_2 (y - y_0) + n_3 (z - z_0) = 0.
  \]

- linear equation of a plane: Substituting \( \mathbf{n} = [n_1, n_2, n_3] \), \( \mathbf{r} = [x, y, z] \), and \( \mathbf{r}_0 = [x_0, y_0, z_0] \) into the second vector equation of a plane yields
  \[
  n_1 x + n_2 y + n_3 z = n_1 x_0 + n_2 y_0 + n_3 z_0.
  \]

Accordingly, given a linear equation of a plane, we can immediately read off a normal vector for the plane: \( \mathbf{n} = [n_1, n_2, n_3] \). For instance, in s681x14 (our first MATLAB example), a normal vector for the plane \( 2x - y + z = 1 \) is \( \mathbf{n} = [2, -1, 1] \), whose components are the respective \( x \)-, \( y \)-, and \( z \)-coefficients in the linear equation of the plane.

Hand Examples

681/10

Find parametric equations and symmetric equations for the line through the points \( A(2, -7, 5) \) and \( B(-4, 2, 5) \).

Solution

- A direction vector for the line is \( \mathbf{v} = \overrightarrow{AB} = \overrightarrow{B} - \overrightarrow{A} = [-6, 9, 0] \).

- A vector equation for the line is \( \mathbf{r} = \mathbf{L}(t) = \overrightarrow{A} + t\mathbf{v} = [2, -7, 5] + t[-6, 9, 0] \).

- Thus parametric equations for the line are
  \[
  r(t) = [x(t), y(t), z(t)] = \mathbf{L}(t) = [2 - 6t, 9t - 7, 5].
  \]

- Moreover, symmetric equations for the line are
  \[
  \frac{x - 2}{-6} = \frac{y + 7}{9}, \quad z = 5.
  \]

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Find an equation of the plane that passes through the point \( A(-1, -3, 2) \) and contains the line \( \mathbf{L}(t) = [-1 - 2t, 4t, 2 + t] \).

Solution

- (You may presuppose that \( A \) is not on \( \mathbf{L} \). How would you show this?) Recall from high school geometry that three noncollinear points determine a plane. So let’s obtain two points on the line (as position vectors). Any two will do, say \( \mathbf{B} = \mathbf{L}(0) = [-1, 0, 2] \) and \( \mathbf{C} = \mathbf{L}(1) = [-3, 4, 3] \).

- A normal vector to the plane is
  \[
  \mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} i & j & k \\ 0 & 3 & 0 \\ -2 & 7 & 1 \end{vmatrix} = [3, 0, 6].
  \]

- Therefore, an equation of our plane is \( \mathbf{n} \cdot [x, y, z] = \mathbf{n} \cdot \overrightarrow{A} \) or \( 3x + 6z = -3 + 0 + 12 = 9 \). That is, \( 3x + 6z = 9 \) or \( x + 2z = 3 \).

682/46

Determine whether the following two planes are parallel, perpendicular, or neither. If neither, find the angle between them.

- \( P_1 : 2x - 5y + z = 3 \)
- \( P_2 : 4x + 2y + 2z = 1 \)
Solution

The angle \( \theta \) between the planes is the angle between normal vectors to the planes, say \( \mathbf{n}_1 = [2, -5, 1] \) and \( \mathbf{n}_2 = [4, 2, 2] \). Thus

\[
\theta = \cos^{-1} \left( \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{\|\mathbf{n}_1\| \|\mathbf{n}_2\|} \right) = \cos^{-1} \left( \frac{8 - 10 + 2}{\|\mathbf{n}_1\| \|\mathbf{n}_2\|} \right) = \cos^{-1} \left( \frac{0}{2} \right) = \frac{\pi}{2}
\]

Therefore, the planes are perpendicular. (Alternatively, just note that \( \mathbf{n}_1 \cdot \mathbf{n}_2 = 0 \).

MATLAB Examples

s681x14

(a) Find parametric equations for the line through \( A(5, 1, 0) \) that is perpendicular to the plane \( 2x - y + z = 1 \).

(b) At which points does this line intersect the coordinate planes?

Solution

(a) (b) A normal vector to the plane is \( \mathbf{n} = [2, -1, 1] \), as noted in the last bullet of the Summary, \( q.v \). Here is a diary file showing that parametric equations for the line are

\[
r(t) = [x(t), y(t), z(t)] = L(t) = [5 + 2t, 1 - t, t].
\]

It also shows that the line intersects the coordinate planes at the following points. In each case, we find the value of \( t \) at which the intercept occurs, then evaluate \( x, y, z \) via substitution.

- \( xy \)-intercept: \( (5, 1, 0) \)
- \( yz \)-intercept: \( (0, \frac{7}{2}, -\frac{5}{2}) \)
- \( xz \)-intercept: \( (7, 0, 1) \).

Determine whether the lines \( \mathbf{L}_1 \) and \( \mathbf{L}_2 \) are parallel, intersecting, or skew (neither). If they intersect, find their point of intersection.

\[ L_1 : \begin{cases} x - 1 \over 2 = s \\ y \over 1 = z - 1 \over 4 = s \\ \end{cases} \]

\[ L_2 : \begin{cases} x \over 1 = y + 2 \over 2 = z + 2 \over 3 = t \\ \end{cases} \]

Solution

Stewart has given symmetric equations for the line; this won’t do. We prefer vector/parametric equations. Use different parameters for the lines, as shown above! (REASON: The lines may pass through the same intersection point, but at different “times.” ANALOGY: We both ate at the Carnegie Deli in New York City, but on different days.)

- \( L_1(s) = [2s + 1, s, 4s + 1] \)
- \( L_2(t) = [t, 2t - 2, 3t - 2] \)

Now set these vectors equal and simultaneously solve for \( s \) and \( t \). We see that there is a solution and hence a point at which the lines intersect. Indeed, when \( s = 0 \) and \( t = 1 \), we have

\[ L_1(0) = L_2(1) = [1, 0, 1]. \]
\[L_1 = [2s+1, s, 4s+1];
L_2 = [t, 2t-2, 3t-2];
e = L_1 - L_2\]
\[e = [2s+1-t, s-2t+2, 4s+3-3t]\]
\[\begin{bmatrix} s \\ t \end{bmatrix} = \text{solve}(e(1), e(2), e(3))\]
\[\text{Warning: 3 equations in 2 variables.}\]
\[s = 0\]
\[t = 1\]
\[
\text{int}_pt = \text{subs}(L_1)
\text{int}_pt = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}
\text{check} = \text{subs}(L_2)
\text{check} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}
\]
\% echo off; diary off

\textbf{s682x52}

Find an equation of the plane consisting of all points that are equidistant from the points \(A(-4, 2, 1)\) and \(B(2, -4, 3)\).

\textbf{Solution}

- The midpoint of the line segment between \(A\) and \(B\) is certainly a point on the plane. Its position vector is 
  \[\overrightarrow{M} = \frac{1}{2}(\overrightarrow{A} + \overrightarrow{B}) = [-1, -1, 2].\]

- A normal vector to the plane is \(n = \overrightarrow{AB} = [6, -6, 2]\).

- Hence an equation of the plane is \(n \cdot [x, y, z] = n \cdot \overrightarrow{M}\) or 
  \(6x - 6y + 2z = 4\) or \(3x - 3y + z = 2\).

\% Stewart 682/52
\%
syms x y z
A = [-4 2 1]; B = [2 -4 3];
M = (A + B) / 2
M = [-1 -1 2]
n = B - A
n = [6 -6 2]
eq0 = dot(n, [x y z]) - dot(n, M);
pretty(eq0)
\[6x - 6y + 2z - 4\]
\% Recall that eq0 is an expression understood to be set to 0. Therefore, an equation of the plane is given by
\%
\% echo off; diary off

\textbf{s682x64}

Find the distance from the point \(A(3, -2, 7)\) to the plane 
\[4x - 6y + z = 5.\]

\textbf{Solution}

There are many ways to compute this distance \(d\). One way is to select any point on the plane—say \(C(1, 0, 1)\)—then compute the length of the vector projection of \(\overrightarrow{AC}\) onto a normal vector 
\[n = [4, -6, 1].\] (Draw a picture!) This gives \(d = \frac{26}{\sqrt{53}} \approx 3.57.\)

\% Stewart 682/64
\%
A = sym([3 -2 7]); n = sym([4 -6 1]); C = sym([1 0 1]);
AC = C - A
AC = [-2, 2, -6]
v = proj(n, AC)
v = [-104/53, 156/53, -26/53]
d = len(v); pretty(d)
\[26 \quad 1/2\]
\[-\quad 53\]
\[53\]
double(d)
ans = 3.5714
\% echo off; diary off