1. Consider the series \( \sum_{n=1}^{\infty} \frac{1}{5+n^2} \).

(a) Use the Comparison Theorem with a \( p \)-series to show that our series converges.

- Let \( a_n = \frac{1}{5+n^2} \) and \( b_n = \frac{1}{n^2} \). Then we note 0 < \( a_n < b_n \) and \( \sum b_n \) converges (\( p \)-series with \( p = 5 > 1 \)). So our series converges by the Comparison Test (CT).

(b) Use the sum of the first ten terms to approximate the sum of our series as a decimal.

- We have \( s_{10} = \sum_{n=1}^{10} \frac{1}{5+n^2} \approx 0.1993 \).

(c) Estimate the error by computing an integral with the \( p \)-series.

- The Remainder Estimate for the Integral Test gives

\[
|\text{error}| < \int_{10}^{\infty} \frac{1}{5+x^2} \, dx < \int_{10}^{\infty} x^{-5} \, dx
\]

\[
< \lim_{b \to \infty} \left( \frac{1}{4x^4} \right)_{10}^b = \lim_{b \to \infty} \left( -\frac{1}{4b^4} + \frac{1}{40000} \right) = \frac{1}{40000} = 2.5 \times 10^{-5}.
\]

2. Consider the series \( \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^6} \).

(a) Use an appropriate test to show that it converges.

- Now \( b_n = |a_n| = \frac{1}{n^6} \downarrow 0 \). Hence the series converges by the Alternating Series Test (AST).

(b) How many terms of the series do we need to add in order to find the sum with \( |\text{error}| < 5 \times 10^{-5} \)?

- By the Alternating Series Estimation Theorem (ASET), the error in using the \( n \)th partial sum \( s_n \) satisfies

\[
|\text{error}| \leq |a_{n+1}| < 5 \times 10^{-5}
\]

\[
\frac{1}{(n+1)^6} < \frac{5}{10^5}
\]

\[
n > \sqrt[6]{20000} - 1 \approx 4.21.
\]

Choose \( n = 5 \).

(c) Compute this estimate as a number with 4 decimal places.

- We have \( s_5 = \sum_{n=1}^{5} \frac{(-1)^{n+1}}{n^6} \approx 0.9856 \).

3. Use appropriate test(s) to determine whether the series \( \sum_{n=1}^{\infty} \frac{1-n}{2+3n} \) is absolutely convergent, conditionally convergent, or divergent.

- As \( n \to \infty \), the Root Test yields

\[
\sqrt[n]{|a_n|} = \frac{n-1}{3n+2} = \frac{1-\frac{1}{n}}{3+\frac{2}{n}} \to \frac{1}{3} < 1.
\]

The series is absolutely convergent.

4. Consider the power series \( \sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1} \frac{1}{2^n} (x-1)^n \).

(a) Find its radius of convergence \( R \).

- Cite an appropriate test.

  - The Root Test with Generalized Fun Fact (GFF) gives \( \sqrt[n]{|a_n|} = \frac{|x-1|}{2^n} \to \frac{|x-1|}{2} \). We need \( \frac{|x-1|}{2} < 1 \) or \( |x-1| < 2 \), which implies \( -1 < x < 3 \): the distance of \( x \) from the center \( a = 1 \) is less than \( R = 2 \), the radius of convergence.

(b) Find its interval of convergence \( I \).

- Cite appropriate test(s).

  - At \( x = -1 \), \( \sum a_n = \sum \frac{1}{2n-1} \), which diverges via the Limit Comparison Test with \( b_n = \frac{1}{n} \). For \( \frac{a_n}{b_n} = \frac{n}{2n-1} \to \frac{1}{2} \) and \( \sum b_n \) is the divergent harmonic series. So \( \sum a_n \) also diverges.

  - At \( x = 3 \), \( \sum a_n = \sum \frac{(-1)^n}{2n-1} \), which converges by the Alternating Series Test (AST) since \( b_n = |a_n| = \frac{1}{2n-1} \downarrow 0 \).

  - Therefore, the interval of convergence is \( I = (-1, 3] \).

Notes