Spring 2008 Math 152/STEPS
Series Supplement 1
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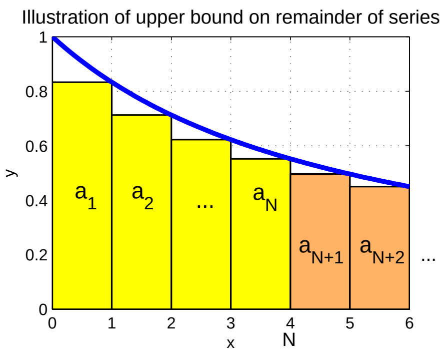
Pix that illustrate bounds on R_n

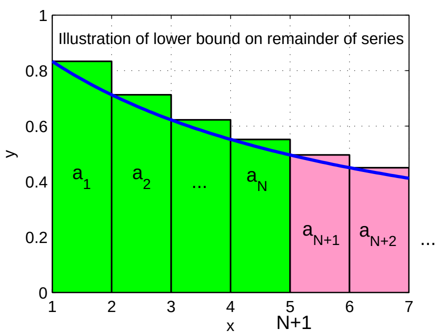
Suppose that $\sum_{n=1}^{\infty} a_n$ is a series of positive terms that converges by the Integral Test to the sum s. Associated with the N-th partial sum $s_N = \sum_{n=1}^{N} a_n$ of the series is the N-th remainder given by

$$R_N = s - s_n = \sum_{n=N+1}^{\infty} a_n.$$

Here are plots which illustrate that lower and upper bounds for R_n are given by the left and right integrals in the following inequality, where f is continuous, decreasing, and integrable on $[1, \infty)$ with $f(n) = a_n$ for positive integers n.

$$\int_{N+1}^{\infty} f(x)\,dx \le R_n = \sum_{n=N+1}^{\infty} a_n \le \int_{N}^{\infty} f(x)\,dx$$





If we add $s_n = \sum_{n=1}^{N} a_n$ to the aforementioned inequality, we obtain

$$P = s_n + \int_{N+1}^{\infty} f(x)\,dx \le s \le s_n + \int_{N}^{\infty} f(x)\,dx = Q.$$

An even closer estimate of s is given by $\frac{1}{2}(P + Q)$. (Just draw a number line to convince yourself of this.)

The Generalized Fun Fact (GFF)

Let $p(n) = \sum_{n=0}^{m} c_k n^k$ be a polynomial in n with $c_m > 0$. Then $\lim_{n\to\infty} \sqrt[n]{p(n)} = 1$. (This increases applicability of the Root Test.)

Proof

We'll prove the fact in four stages, the first three of which use the same technique from Section 4.8 in Calc 1 to compute a limit.

1. $\lim_{n\to\infty} \sqrt[n]{c_m} = 1$
   - Let $y = (c_m)^{1/n}$. Then $\lim_{n\to\infty} \ln y = \lim_{n\to\infty} \frac{\ln c_m}{n} = 0$, whence $\lim_{n\to\infty} y = \lim_{n\to\infty} e^{\ln y} = e^0 = 1$.
2. $\lim_{n\to\infty} \sqrt[n]{n} = 1$
   - Let $y = n^{1/n}$. Then
   $$\lim_{n\to\infty} \ln y = \lim_{n\to\infty} \frac{\ln n}{n} \overset{\text{L'H}}{=} \lim_{n\to\infty} \frac{1/n}{1} = 0,$$
   whence $\lim_{n\to\infty} y = \lim_{n\to\infty} e^{\ln y} = e^0 = 1$.
3. $\lim_{n\to\infty} \sqrt[n]{1 + \sum_{k=0}^{m-1} \frac{c_k/c_m}{n^{m-k}}} = 1$
   - Let $y = \left(1 + \sum_{k=0}^{m-1} \frac{c_k/c_m}{n^{m-k}}\right)^{1/n}$. Then
   $$\lim_{n\to\infty} \ln y = \lim_{n\to\infty} \frac{\ln\left(1 + \sum_{k=0}^{m-1} \frac{c_k/c_m}{n^{m-k}}\right)}{n} = 0,$$
   whence $\lim_{n\to\infty} y = \lim_{n\to\infty} e^{\ln y} = e^0 = 1$.
4. $\lim_{n\to\infty} \sqrt[n]{p(n)} = \lim_{n\to\infty} \sqrt[n]{\sum_{n=0}^{m} c_k n^k} = 1$
   - We use 1–3 above to dispatch this limit.
   $$\begin{aligned} &\lim_{n\to\infty} \sqrt[n]{\sum_{n=0}^{m} c_k n^k} \\ =\;& \lim_{n\to\infty} \left(\sqrt[n]{c_m n^m}\ \sqrt[n]{1 + \sum_{k=0}^{m-1} \frac{c_k/c_m}{n^{m-k}}}\right) \\ =\;& \lim_{n\to\infty} \left(\sqrt[n]{c_m}\ \left(\sqrt[n]{n}\right)^m\ \sqrt[n]{1 + \sum_{k=0}^{m-1} \frac{c_k/c_m}{n^{m-k}}}\right) \\ =\;& (1)\,(1)^m\,(1) = 1 \end{aligned}$$