10.2 Calculus with Parametric Curves

1. [655/3] Find an equation of the tangent line to the curve $x = t^3 + 1, y = t^4 + t$, for $t = -1$.

Now $dx/dt = 3t^2$ and $dy/dt = 4t^3 + 1$. The derivative is $dy/dx = dy/dt \cdot dt/dx = 4t^3 + 1 / 3t^2$. For $t = -1, m = dy/dx = -1$ at point $(x, y) = (0, 0)$. The tangent line is $y = -x$. Here’s a plot.

2. [655/7] Find parametric and Cartesian equations for the tangent line to the curve $x = t + \ln t, y = t^2 + 2$, at $(1, 3)$.

So $dx/dt = 1/ t$ & $dy/dt = 2t$. When $t = 1, (x, y) = (1, 3)$. A direction vector for the tangent line is $v = [dx/dt, dy/dt]_{t=1} = [1, 2]$. Thus $[x, y] = L(t) = [1, 3] + t[1, 2]$ or $x = t + 1, y = 2t + 3$ give parametric equations for the tangent line. Now $t = x - 1$, whence a Cartesian equation of the tangent line is given by $y = 2(x - 1) + 3$ or $y = 2x + 1$.

3. [655//11] Find $dy/dx$ and $d^2y/dx^2$ given $x = t^2 + 1, y = t^2 + t$. For which values of $t$ is the curve concave upward?

So $dx/dt = 2t & dy/dt = 2t + 1$ and $dy/dx = dy/dt \cdot dt/dx = 1 + \frac{1}{2}t^{-1}$.

Now $d^2y/dx^2 = d(dy/dx) / dt = -\frac{1}{4}t^{-2}$. Therefore $d^2y/dx^2 = \frac{dy}{dx} = -\frac{1}{4}t^{-2} / 2t = -\frac{1}{4t^3} > 0$ for $t < 0$, where the curve is concave upward.

4. [655/17] Find the points on the curve $x = t^3 - 3t, y = t^2 - 3$, where the tangent is horizontal or vertical. Illustrate.

Now $dx/dt = 3t^2 - 3$ and $dy/dt = 2t$. We’ll examine horizontal tangents first, then vertical ones, and finally produce a plot.

- Now $dy/dx = dy/dt \cdot dt/dx = 0$ if $dx/dt = 0 \& dy/dt \neq 0$. Solve $dx/dt = 2t = 0$ to obtain $t = 0$ and $dy/dt = -3 \neq 0$. So for $t = 0$, the curve has a horizontal tangent at $(x, y) = (0, -3)$.

- Next $dy/dx = dy/dt / dx/dt = \pm \infty$ if $dx/dt = 0 \& dy/dt \neq 0$. Solve $dx/dt = 3t^2 - 3 = 0$ to obtain $t = \pm 1$ and $dy/dt = \pm 2 \neq 0$. So for $t = \pm 1$, the curve has a horizontal tangent at $(x, y) = (\mp 2, -2)$.

- Here is an illustrative plot of curve, points, & tangents.

5. [655/26] Graph the curve $x = -2 \cos t, y = \sin t + \sin 2t$ to discover where the curve crosses itself. Then find equations of both tangents at that point. Illustrate with a graph.

From a graph (see below), the curve crosses itself at $(x, y) = (1, 0), corresponding to $t = \frac{\pi}{2}$ or $t = \frac{3\pi}{2}$. Let $r(t) = [x(t), y(t)]$. Then $r'(t) = [x'(t), y'(t)]$.

Let $v_1 = r'(\frac{\pi}{2}) = [\sqrt{3}, -\frac{1}{2}]$, $v_2 = r'(\frac{3\pi}{2}) = [-\sqrt{3}, -\frac{1}{2}]$.

With $P = [1, 0]$, we have the following tangent lines.

$L_1(t) = P + tv_1 = \begin{bmatrix} 1 + \sqrt{3}t, -\frac{3}{2}t \end{bmatrix}$

$L_2(t) = P + tv_2 = \begin{bmatrix} 1 - \sqrt{3}t, -\frac{3}{2}t \end{bmatrix}$
6. [655/32] Find the area enclosed by the curve \( x = t^2 - 2t \), \( y = \sqrt{t} \) and the y-axis.

\[
\int_{t_1}^{t_2} \left| \frac{dy}{dt} \right| \, dt
\]

When the curve intersects the y-axis, \( x = t (t - 2) = 0 \), so \( t = 0, 2 \). The area \( A \) between the curve and the y-axis is
\[
A = \int_0^2 \left| \frac{dy}{dt} \right| \, dt = \int_0^2 \, dt = 2.
\]

7. [656/46] Graph the curve \( x = \cos t + \ln(\tan \frac{1}{2}t) \), \( y = \sin t \), \( \frac{\pi}{4} \leq t \leq \frac{3\pi}{4} \) and find its arc length.

The arc length is
\[
L = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \sqrt{\left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2} \, dt.
\]

8. [656/48] Find the length of the loop of the curve \( x = 3t - t^3 \), \( y = 3t^2 \).

From the graph, the loop closes at the y-axis, where \( x = 0 \). Solve \( x = 3t - t^3 = 0 \) to obtain \( t = 0, \pm \sqrt{3} \). The loop is traced out in a counterclockwise fashion as \( t \) increases from \( -\sqrt{3} \) to \( \sqrt{3} \). The arc length is
\[
L = \int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{\left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2} \, dt
\]

9. [656/62] Find the surface area obtained by rotating the curve \( x = 2r^2 + \frac{1}{r} \), \( y = 8\sqrt{t} \), \( 1 \leq t \leq 3 \), about the x-axis.

The surface area is \( S = \int_{1}^{3} 2\pi ry \, ds \), where
\[
ds = \sqrt{\left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2} \, dt.
\]

Using a CAS, we have
\[
S = \int_{1}^{3} 2\pi (8\sqrt{t}) \left( \frac{1}{4} + \frac{1}{t^2} \right) \, dt = \frac{2\pi}{15} (103\sqrt{3} + 3) \approx 1215.76 \text{cm}^2.
\]

10. [657/66] Find the surface area obtained by rotating the curve \( x = e^t - t \), \( y = 4e^{t/2} \), \( 0 \leq t \leq 1 \), about the y-axis.

The surface area is \( S = \int_{0}^{1} 2\pi rx \, ds \), where
\[
ds = \sqrt{\left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2} \, dt.
\]

Using a CAS, we have
\[
S = \int_{0}^{1} 2\pi (e^t - t) (e^t + 1) \, dt = (e^2 + 2e - 6) \pi \approx 21.4433 \text{cm}^2.
\]