Spring 2007 Math 152
Exam 2: Executive Summary
Fri, 09/Mar ©2007, Art Belmonte

Chapters and Sections

8: Techniques of Integration

8.8 Approximate Integration

8.9 Improper Integrals

9: Further Applications of Integration

9.1 [Separable First-Order] Differential Equations

9.2 First-Order Linear Equations

9.3 Arc Length

9.4 Area of a Surface of Revolution

9.5 Moments and Centers of Mass

9.6 Hydrostatic Pressure and Force

Fundamental Concepts

Legend

$h = \Delta x = \frac{b-a}{n}$ $\quad dA = dx\,dy = dy\,dx$ $\quad K = \max_{a \le x \le b} \left| f''(x) \right|$

$x_k = a + kh;\ k = 0, \ldots, n$ $\quad z$: depth $\quad M = \max_{a \le x \le b} \left| f^{(4)}(x) \right|$

8.8

$$L_n = h \sum_{k=0}^{n-1} f(x_k) \quad R_n = h \sum_{k=1}^{n} f(x_k)$$

$$M_n = h \sum_{k=1}^{n} f\left(\frac{x_{k-1} + x_k}{2}\right)$$

$$T_n = h \left(\frac{f(x_0) + f(x_n)}{2} + \sum_{k=1}^{n-1} f(x_k) \right)$$

$$S_n = \frac{h}{3} \left(f(x_0) + f(x_n) + 4 \sum_{i=0}^{n/2-1} f(x_{2i+1}) + 2 \sum_{j=1}^{n/2-1} f(x_{2j}) \right)$$

$$\left| E_{T_n} \right| \le \frac{K(b-a)^3}{12n^2} \qquad \left| E_{M_n} \right| \le \frac{K(b-a)^3}{24n^2}$$

$$\left| E_{S_n} \right| \le \frac{M(b-a)^5}{180n^4}$$ [For Simpson's Rule, n is even!]

8.9 Use *limits* and/or comparison theorems to compute improper integrals.

9.1 $dy/dx = f(x,y) \Longrightarrow g(y)\,dy = h(x)\,dx \Longrightarrow G(y) = H(x) + C \Longrightarrow y = G^{-1}(H(x) + C)$

9.2

$$y' + p(x)\,y = q(x)$$
$$\mu(x) = \exp\left(\int p(x)\,dx\right)$$
$$\mu(x)\,y' + \mu(x)\,p(x)\,y = \mu(x)\,q(x)$$
$$(\mu(x)\,y)' = \mu(x)\,q(x)$$
$$y = \frac{1}{\mu(x)}\left(\int \mu(x)\,q(x)\,dx + C\right)$$

9.3

$$L = \int ds \qquad ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

9.4 $S = \int 2\pi r\,ds$

9.5 Let $\mathbf{p} = \begin{bmatrix} m_1 & m_2 & \cdots & m_n \end{bmatrix}$, $m = \sum_{k=1}^{n} m_k$, and

$$\mathbf{r} = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix}.$$ Then $\mathrm{CM} = [\bar{x}, \bar{y}] = \frac{1}{m}\,\mathbf{pr}$.

Let a planar region D have density δ. Then its mass is $m = \iint_D \delta\,dA$ and its center of mass is given by $\mathrm{CM} = [\bar{x}, \bar{y}] = \frac{1}{m} \iint_D \delta\,[x,y]\,dA$.

9.6 $P = \delta z = \rho g z,\ dA = w\,dy,\ dF = P\,dA,\ F = \int dF$

Additional Details

- $\displaystyle\int_a^\infty \frac{1}{x^p}\,dx$ converges for $p > 1$ and diverges for $p \le 1$.
- Comparison theorems for improper integrals: Let f and g be continuous on (a, ∞) with $f \ge g \ge 0$. [Just think of areas under curves above the x-axis and the following assertions are clear.]
  1. If $\int_a^\infty f(x)\,dx$ converges, then so does $\int_a^\infty g(x)\,dx$. Moreover, $0 \le \int_a^\infty g(x)\,dx \le \int_a^\infty f(x)\,dx = L$.
  2. If $\int_a^\infty g(x)\,dx$ diverges, then so does $\int_a^\infty f(x)\,dx$. Moreover, $\int_a^\infty g(x)\,dx = \int_a^\infty f(x)\,dx = \infty$.
- If a differential equation has an initial condition, $y(x_0) = y_0$, then this initial condition is used to resolve the constant involved in the general solution to the differential equation using algebra.

General framework for an [improper] integral

Let $(a,b) \subset \mathbb{R}$. Here a is either a real number or $-\infty$ and b is either a real number or ∞. Label $x_0 = a$ and $x_m = b$. Let $\{x_i\}_{i=1}^{m-1}$ be real numbers such that $x_0 < x_1 < \cdots < x_m$ and define $D = (a,b) \setminus \{x_i\}_{i=1}^{m-1}$. [In other words, D is (a,b) with the points $x_1, \ldots, x_{m-1}$ removed.] Suppose that $f : D \to \mathbb{R}$ is continuous and that f is discontinuous at $x_1, \ldots, x_{m-1}$. Finally, let $c_i \in (x_{i-1}, x_i)$, $i = 1, \ldots, m$. [Note that f is continuous at the interior points c_i.] Then we define

$$\int_a^b f(x)\,dx = \sum_{i=1}^{m} \left(\lim_{t \to x_{i-1}^+} \int_t^{c_i} f(x)\,dx + \lim_{t \to x_i^-} \int_{c_i}^{t} f(x)\,dx \right),$$

provided that *all* these limits exist.