Spring 2008 Math 152
Exam 3: Executive Summary
Fri, 18/Apr ©2008, Art Belmonte

Chapters and Sections: 10.1–10.9 + 11.1–11.2

10.1

- **Sequence**: $\{a_1, a_2, a_3, \dots\} = \{a_n\}_{n=1}^{\infty}$
- **GST**: For $|r| < 1$, $r^n \to 0$ (as $n \to \infty$)
- **MST**: $a_n \uparrow$ or $a_n \downarrow$ & $|a_n| \le M \implies a_n \to L$
- For a_n **recursive** satisfying MST, let $a_{n+p} \to L$ in formula then solve for L and select choice.

10.2

- **Series**: $a_1 + a_2 + a_3 + \cdots = \sum_{n=1}^{\infty} a_n = \sum a_n$
- Sequence of **partial sums**: $s_n = \sum_{k=1}^{n} a_k$
- **Convergent** series: If $s_n \to s$, then $s = \sum a_n$ is the sum; otherwise, the series **diverges**.
- **TD**: If $\lim a_n \ne 0$, then $\sum a_n$ diverges.
- **GST**: For $|r| < 1$, $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$; a: first term; r: ratio.
- **Telescoping** series: Use partial fractions for partial sums.

10.3

- **IT**: If $a_n = f(n)$ where $f > 0$ is con't and $\downarrow$, then $\sum a_n$ and $\int_N^{\infty} f$ either both converge or both diverge.
- **CT**
  - $0 < a_n \le b_n$ & $\sum b_n$ conv $\implies \sum a_n$ conv
  - $0 < b_n \le a_n$ & $\sum b_n$ divg $\implies \sum a_n$ divg
- **LCT**: Let $\lim\sup \frac{a_n}{b_n} = c$ where $a_n, b_n > 0$.
  - If $c = 0$ & $\sum b_n$ converges, then $\sum a_n$ converges.
  - If $c > 0$ is finite, then $\sum a_n$ and $\sum b_n$ either both converge or both diverge.
  - If $c = \infty$ & $\sum b_n$ diverges, then $\sum a_n$ diverges.
- **Remainder**: $R_n = s - s_n = \sum_{k=n+1}^{\infty} a_k$.

$$\int_{n+1}^{\infty} f(x)\,dx \le R_n \le \int_n^{\infty} f(x)\,dx$$

$$L = s_n + \int_{n+1}^{\infty} f(x)\,dx \le s \le s_n + \int_n^{\infty} f(x)\,dx = U$$

$$\text{better approximation} = \tfrac{1}{2}(L + U)$$

10.4

- **Alternating series**: $\sum (-1)^n b_n$ where $b_n = |a_n|$
- **AST**: If $b_n \downarrow 0$, then $\sum a_n$ converges.
- **ASET**: $|R_n| = |a_{n+1}|$
- **Absolutely convergent**: $\sum |a_n|$ converges.
- **Conditionally convergent**: $\sum a_n$ converges, but $\sum |a_n|$ diverges.
- Let $\lim\sup \left|\frac{a_{n+1}}{a_n}\right| = L$ (**Ratio Test**) or $\lim\sup \sqrt[n]{|a_n|} = L$ (**Root Test**).
  - If $L < 1$, then $\sum a_n$ is absolutely convergent.
  - If $L > 1$, then $\sum a_n$ diverges.
  - If $L = 1$, test is inconclusive. Use other tests!
- **GFF**: $\sqrt[n]{p(n)} \to 1$ for polynomials p with pos leading coeff.

10.5

- **Power series**: $\sum_{n=0}^{\infty} c_n (x-a)^n$
- **Radius of convergence *R***: series converges for $|x-a| < R$ and diverges for $|x-a| > R$. Use Ratio or Root test to determine R.
- Use other tests to determine if one, both, or neither endpoints $x = a \pm R$ are included in the **interval of convergence *I***.

10.6

- A power series may be differentiated or integrated term-by-term within its radius of convergence R. The resulting power series has the same radius of convergence R (though perhaps not the same interval of convergence).
- Recall the GST: $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$ for $|z| < 1$. Use this along with algebraic manipulation, differentiation, and/or integration to obtain power series representations.

10.7

- Taylor series for f: $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$
- Remainder $R_n = \sum_{k=n+1}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$ satisfies $|R_n| \le \frac{M|x-a|^{n+1}}{(n+1)!}$ for $|x-a| < R$ where $M = \sup_{|x-a|<R} \left| f^{(n+1)}(x) \right|$. (Think of sup as max.)
- See reverse for important Maclaurin series ($a = 0$).

10.8

- **Binomial series**: For $|z| < 1$ and $m \in \mathbb{R}$,

$$
\begin{aligned}
& (1+z)^m \\
= \; & \sum_{k=0}^{\infty} \binom{m}{k} z^k \\
= \; & 1 + mz + \frac{m\,(m-1)}{2!} z^2 + \frac{m\,(m-1)\,(m-2)}{3!} z^3 + \cdots
\end{aligned}
$$

10.9

- n^{th}-degree Taylor polynomial: $\sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} (x-a)^k$

11.1

- **Distance** from P to Q: $\|Q - P\| = \sqrt{\sum_{k=1}^{n} (q_k - p_k)^2}$
- **Hypersphere**: $\sum_{k=1}^{n} (x_k - c_k)^2 = r^2$

11.2

- $\mathbf{a} \cdot \mathbf{b} = \sum_{k=1}^{n} a_k b_k = \|\mathbf{a}\| \, \|\mathbf{b}\| \cos\theta$
- $\operatorname{comp}_{\mathbf{a}} \mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|}$
- $\operatorname{proj}_{\mathbf{a}} \mathbf{b} = \left(\frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|} \right) \frac{\mathbf{a}}{\|\mathbf{a}\|}$
- Plus the Usual Suspects. . .

Important Maclaurin series and their radii of convergence

- $\displaystyle \sum_{n=0}^{\infty} z^n = \frac{1}{1-z}, \; R = 1$
- $\displaystyle e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \; R = \infty$
- $\displaystyle \sin z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}, \; R = \infty$
- $\displaystyle \cos z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n}, \; R = \infty$
- $\displaystyle \ln(1+z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} z^n}{n}, \; R = 1$
- $\displaystyle \tan^{-1} z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{2n+1}, \; R = 1$