Spring 2004 Math 253/501–503
12 Multivariable Differential Calculus
12.1 Functions of Several Variables
Thu, 29/Jan ©2004, Art Belmonte

Summary

- **function of \(n\) variables**: a rule that assigns a unique real number \(f(x) = f(x_1, \ldots, x_n)\) to each ordered \(n\)-tuple \(x = [x_1, \ldots, x_n]\) in a subset \(D\) (the domain) of \(\mathbb{R}^n\).

- **plots of \(w = f(x)\)**:
  - For \(n = 1\), these are curves \(y = f(x)\) in the \(xy\)-plane.
  - For \(n = 2\), these are surfaces \(z = f(x, y)\) in \(xyz\)-space.
  - For \(n > 2\), these are hypersurfaces \(w = f(x)\) embedded in \((n + 1)\)-space.

- **level hypersurface of \(f\)**: where \(f(x) = k\), a constant
  - For \(n = 2\), the graphs of \(f(x, y) = k\) are called **level curves**.
  - For \(n = 3\), the graphs of \(f(x, y, z) = k\) are called **level surfaces**.

Hand Examples

**730/3**

Let \(F(x, y) = \frac{3xy}{x^2 + 2y^2}\). Evaluate \(F(1, 1)\), \(F(-1, 2)\), \(F(t, 1)\), \(F(-1, y)\), and \(F(x, x^2)\).

Solution

By hand, just plug and chug. With a TI-89 or MATLAB, define a function and blaze away! (I’ll do the TI-89 in class with you. See the corresponding MATLAB example.)

\[
F(1, 1) = \frac{3(1)(1)}{(1)^2 + 2(1)^2} = \frac{3}{3} = 1
\]

\[
F(-1, 2) = \frac{3(-1)(2)}{(-1)^2 + 2(2)^2} = \frac{-6}{9} = -\frac{2}{3}
\]

\[
F(t, 1) = \frac{3(t)(1)}{(t)^2 + 2(1)^2} = \frac{3t}{t^2 + 2}
\]

\[
F(-1, y) = \frac{3(-1)(y)}{(-1)^2 + 2(y)^2} = \frac{-3y}{2y^2 + 1}
\]

\[
F(x, x^2) = \frac{3(x)(x^2)}{(x)^2 + 2(x^2)^2} = \frac{3x^3}{x^2 + 2x^4} = \frac{3x}{1 + 2x^2}
\]

MATLAB Examples

**s730x03 [730/3 revisited]**

Let \(F(x, y) = \frac{3xy}{x^2 + 2y^2}\). Evaluate \(F(1, 1)\), \(F(-1, 2)\), \(F(t, 1)\), \(F(-1, y)\), and \(F(x, x^2)\).
Solution

Here we define $F$ in an external function M-file. Then we use a script M-file to record our work.

```matlab
%====================
function z = F(x,y)
    z = 3*x*y / (x^2 + 2*y^2);
%====================
```

% Stewart 730/3
% format rat
syms t x y
a = F(1,1)
a =

1

b = F(-1,2)
b =

-2/3
c = F(t,1); pretty(c)
c =

\frac{3}{2} \quad t \\
\quad \quad t + 2

d = F(-1,y); pretty(d)
d =

\frac{-3}{2} \quad y \\
\quad \quad \quad \quad 1 + 2 y

e = simple(F(x,x^2)); pretty(e)
e =

\frac{3}{2} \quad x \\
\quad \quad 1 + 2 x
```

% echo off; diary off

s730x38

Plot the function $f(x, y) = \sqrt{16 - x^2 - 16y^2}$.

Solution

We illustrate six ways to do this, discussing the merits and drawbacks of each.

(a) The first way is to draw a contour plot. This is like a topographical map. The idea is to take traces (cross sections) $z = f(x, y) = k$ and project them onto the $xy$-plane to form level curves of the function $f$. The MATLAB contour command does a nice job of this. The drawback is that it’s a 2-D plot. You must imagine levels forming hills or valleys.

(b) The second way is to use the `impl` command mentioned on page 104 of Cooper. This basically raises the contours from (a) up or down in space, giving you a “skeletal” view of the surface—not a real surface, but good enough for government work, I suppose. At least the figure is rendered in 3-D space.

(c) The third way is to use the `ezsurf` command in MATLAB’s Symbolic Math Toolbox. Like an errant youth, surfaces “go bad” when they go vertical, as evidenced by the degradation in the graph near the $xy$-plane. Still, `ezsurf` is easy to use.

(d) A fourth way is to use MATLAB’s `surf` command with a rectangular domain. Again, note the degradation in the graph when the surface becomes vertical.

(e) A fifth way is with my `hvsd` command followed by `surf`. This enables one to use a non-rectangular domain that results in less degradation. Still, you’re trying to fit a rectangular peg into an elliptical hole, with the “pinching” at the boundary that this entails.

(f) Finally, we see the sixth way: parametric surface plotting! This is the real deal from Chapter 14 of Stewart (Section 14.6 in particular). This gives the nicest plot at the expense of having to come up with a surface parameterization, in this case ellipsoidal or scaled spherical coordinates. This is beyond your current knowledge, but we’ll cover it in the fullness of time.

Here are the six sets of diary files and plots.

```matlab
% Stewart 730/38a: Via contour
%x = linspace(-4, 4); y = linspace(-1, 1);
[X,Y] = meshgrid(x,y);
Z = sqrt(16 - X.^2 - 16.*Y.^2);
v = [0.5 1.5 2.5 3.0 3.5 3.8];
[cs, h] = contour(X,Y,Z,v);
clabel(cs,h,v)
% echo off; diary off
```

```matlab
% Stewart 730/38b: Via impl
F = inline('x.^2 + 16.*y.^2 + z.^2', 'x', 'y', 'z');
span = [-4 4 -1 1 0 4];
impl(F, span, 16);
ans =
The max over this domain is 48.00000
ans =
The min over this domain is 0.00000
grid on; axis equal
% echo off; diary off
```
Draw a contour map of the function \( f(x, y) = x^2 - y^2 \).

[Additionally, describe its level curves and sketch the surface.]

**Solution**

The level curves of \( f \) are \( f(x, y) = x^2 - y^2 = k \), where \( k \) is constant. These are hyperbolas (or straight lines if \( k = 0 \)). Here are codes for a contour map and a surface plot, followed by pix.
Stewart 730/44d: contour plot

\[ X = \text{linspace}(-4, 4, 20); \ y = x; \]
\[ [X,Y] = \text{meshgrid}(x,y); \]
\[ Z = X.^2 - Y.^2; \]
\[ \text{surf}(X,Y,Z); \ \text{grid on} \]
\[ \text{echo off; diary off} \]

Stewart 730/44d: surface plot

\[ Z = X.^2 - Y.^2; \]
\[ \text{surf}(X,Y,Z); \ \text{grid on} \]
\[ \text{echo off; diary off} \]