Spring 2004 Math 253/501–503
12 Multivariable Differential Calculus
12.8 Lagrange Multipliers
Thu, 19/Feb ©2004, Art Belmonte

Summary

With \( x = [x_1, \ldots, x_n] \), let \( f(x) \) and \( g_k(x) \), \( k = 1, \ldots, m \), be real-valued functions of \( n \) variables defined on a subset \( D \) of \( \mathbb{R}^n \). To optimize \( f \) (i.e., absolutely maximize or minimize) subject to the \( m \) constraints \( g_k(x) = 0 \), solve the following system of \( m + n \) scalar equations in the \( m + n \) unknowns \( x_1, \ldots, x_n \) and \( \lambda_1, \ldots, \lambda_m \).

\[
\nabla f = \sum_{k=1}^{m} \lambda_k \nabla g_k; \quad g_k(x) = 0, k = 1, \ldots, m
\]

(The first vector equation gives rise to \( n \) scalar equations, one for each slot in the gradient vector.)

- Most frequently for you, \( n = 2 \) and \( m = 1 \). That is, optimize \( f(x, y) \) subject to the single constraint \( g(x, y) = 0 \) by solving \( \nabla f = \lambda \nabla g \) and \( g(x, y) = 0 \) for \( x, y, \lambda \).

- Another common case is when \( n = 3 \) and \( m = 1 \). Optimize \( f(x, y, z) \) subject to the single constraint \( g(x, y, z) = 0 \) by solving \( \nabla f = \lambda \nabla g \) and \( g(x, y, z) = 0 \) for \( x, y, z, \lambda \).

- On occasion, \( n = 3 \) and \( m = 2 \). Optimize \( f(x, y, z) \) subject to the dual constraints \( g(x, y, z) = 0 \) and \( h(x, y, z) = 0 \) by solving \( \nabla f = \lambda \nabla g + \mu \nabla h \) and \( g(x, y, z) = 0 \) and \( h(x, y, z) = 0 \) for \( x, y, z, \lambda, \mu \).

“Hand” Examples

Again, we employ machine power! Use your TI-89 and TAMUCALC when doing problems “by hand.”

788/2

Use Lagrange multipliers to find the maximum and minimum values of \( f(x, y) = 2x + y \) subject to the constraint \( x^2 + 4y^2 = 1 \).

Solution

- Let \( g(x, y) = x^2 + 4y^2 - 1 \). Then \( g(x, y) = 0 \) is our constraint.

- We now solve

\[
\nabla f = \lambda \nabla g \quad \text{and} \quad g = 0
\]

\[
[2, 1] = \lambda [2x, 8y] \quad \text{and} \quad x^2 + 4y^2 - 1 = 0
\]

\[
2 = 2\lambda x, \quad 1 = 8\lambda y, \quad x^2 + 4y^2 = 1
\]

for \( x, y, \lambda \) to obtain \((x, y, \lambda) = \left(\frac{1}{4}\sqrt{17}, \frac{1}{4}\sqrt{17}, \frac{1}{4}\sqrt{17}\right)\) or \((x, y, \lambda) = \left(-\frac{1}{4}\sqrt{17}, -\frac{1}{4}\sqrt{17}, -\frac{1}{4}\sqrt{17}\right)\).

- Crank out function values of \( f \) at these points. DONE!

\[
f\left(\frac{1}{4}\sqrt{17}, \frac{1}{4}\sqrt{17}\right) = \frac{1}{2}\sqrt{17} \approx 2.06, \text{ abs max}
\]

\[
f\left(-\frac{1}{4}\sqrt{17}, -\frac{1}{4}\sqrt{17}\right) = -\frac{1}{2}\sqrt{17} \approx -2.06, \text{ abs min}
\]

- Restrict the domain of \( z = f(x, y) = 2x + y \) (the graph of which is a plane) to the ellipse \( x^2 + 4y^2 = 1 \), a closed, bounded set. Then the restricted function values of \( f \) take on absolute maximum and absolute minimum values. Here is a 3-D picture illustrating this! (Please see the corresponding MATLAB example for the graphics code.)

Recall from Section 12.6 that the gradient vector of a function is perpendicular to level curves of the function. Notice that at the solution points \((x, y)\) obtained above that \( \nabla f \) and \( \nabla g \) are parallel, whence \( \nabla f = \lambda \nabla g \), as predicted by our analytical work! Here’s a contour plot illustrating this.
MATLAB Examples

s788x02 [788/2 revisited]

Use Lagrange multipliers to find the maximum and minimum values of \( f(x, y) = 2x + y \) subject to the constraint \( x^2 + 4y^2 = 1 \).

Solution

Here we replicate the symbolic work we did with our TI-89.

```matlab
%% Stewart 788/2

% Define functional expressions.
syms L x y; v = [x y]; f = 2*x + y; g = x^2 + 4*y^2 - 1;
grad_f = grad(f,v); grad_g = grad(g,v);

% Set up and solve equations.
eq = grad_f - L*grad_g; eq = [eq g];
c = solve(eq(1), eq(2), eq(3));
c = [c.x c.y]

% Crank out function values of f.
echo off; diary off
for k = 1:size(c,1)
p = c(k,:)
f_val = subs(f, [x y], p)
end
```

Here are the codes that produced the figures in the corresponding hand example. First the 3-D plot code.

```matlab
%% Stewart 788/2g

t = linspace(0, 2*pi, 37);
x = cos(t); y = sin(t)/2;
z = 2*x + y;
plot3(x,y,z, 'g', 'LineWidth', 3)
hold on
z = 0*z - 5;
plot3(x,y,z, 'b', 'LineWidth', 3)

x = linspace(-2, 2); y = linspace(-1, 1);
[X,Y] = meshgrid(x,y);
f = 2*X + Y;
grid on
echo off; diary off
```

Now the contour plot code.

```matlab
%% Stewart 788/2c

x = linspace(-2, 2); y = linspace(-1, 1);
[X,Y] = meshgrid(x,y);
f = 2*X + Y;
g = X^2 + 4*Y^2 - 1;
contour(X,Y,f, 30, 'r'); hold on
contour(X,Y,g, [0 0], 'b')

echo off; diary off
```

s788x14

Use Lagrange multipliers to find the maximum and minimum values of \( f(x, y, z) = 3x - y - 3z \) subject to the constraints \( x + y - z = 0 \) and \( x^2 + 2z^2 = 1 \).

Solution

- Let \( g(x, y, z) = x + y - z \) and \( h(x, y, z) = x^2 + 2z^2 - 1 \).
  Then \( g(x, y, z) = 0 \) and \( h(x, y, z) = 0 \) are our constraints. (NOTE: Even though \( y \) does not appear in the functional expression for \( h \), we need to regard \( h \) as a function of three variables so that the gradient slots line up below!)
- We now solve \( \nabla f = \lambda \nabla g + \mu \nabla h \) and \( g(x, y, z) = 0 \) and \( h(x, y, z) = 0 \) for \( x, y, z, \lambda, \mu \) via MATLAB. The absolute maximum of \( f \) subject to the two constraints is
  \[
  f \left( \frac{\sqrt{6}}{3}, -\frac{\sqrt{6}}{3}, -\frac{\sqrt{6}}{3} \right) = 2\sqrt{6} \approx 4.90,
  \]
whereas the absolute minimum is
\[ f \left( -\frac{1}{3}\sqrt{6}, \frac{1}{2}\sqrt{6}, \frac{1}{6}\sqrt{6} \right) = -2\sqrt{6} \approx -4.90. \]

- Can you ascribe a geometrical meaning to this problem like we did in 788/2?

```matlab
% Stewart 788/14
% Define functional expressions.
syms L M x y z; v = [x y z];
f = 3*x - y - 3*z;
g = x + y - z;
h = x^2 + 2*z^2 - 1;
grad_f = grad(f,v)
grad_f = [3, -1, -3]
grad_g = grad(g,v)
grad_g = [1, 1, -1]
grad_h = grad(h,v)
grad_h = [2*x, 0, 4*z]
% Set up and solve equations.
eq = grad_f - (L*grad_g + M*grad_h)
eq = [3-L-2*M*x, -1-L, -3+L-4*M*z]
eq = [eq g h]
eq = [3-L-2*M*x, -1-L, -3+L-4*M*z, x+y-z, x^2+2*z^2-1]
c = solve(eq(1), eq(2), eq(3), eq(4), eq(5))
c = [L: [2x1 sym]
M: [2x1 sym]
x: [2x1 sym]
y: [2x1 sym]
z: [2x1 sym]
c = [c.x c.y c.z]
c = [-1/3*6^6(1/2), 1/2*6^6(1/2), 1/6*6^6(1/2)]
% Crank out fuctions values of f.
echo off
p = [-1/3*6^6(1/2), 1/2*6^6(1/2), 1/6*6^6(1/2)]
func_val = -2*6^6(1/2)
p = [1/3*6^6(1/2), -1/2*6^6(1/2), -1/6*6^6(1/2)]
f789x20
Find the maximum and minimum volumes of a rectangular box whose surface area is 1500 cm² and whose total edge length is 200 cm.

Solution
Draw a diagram. Let x, y, and z be the length, width, and height of the box, respectively. The volume of the box is \( f(x, y, z) = xyz \), whereas its surface area is \( 2xy + 2yz + 2xz = 1500 \) and total edge length is \( 4x + 4y + 4z = 200 \).

- Let \( g(x, y, z) = 2xy + 2yz + 2xz - 1500 \) and \( h(x, y, z) = 4x + 4y + 4z - 200 \). Then \( g(x, y, z) = 0 \) and \( h(x, y, z) = 0 \) are our constraints.

- Solving \( \nabla f = \lambda \nabla g + \mu \nabla h \) and \( g(x, y, z) = 0 \) and \( h(x, y, z) = 0 \) for \( x, y, z, \lambda, \mu \), we obtain six solutions.

- The minimum volume of 2947.94 cm³ occurs when the dimensions of the box (in cm) are
  \[ 6.13 \times 21.94 \times 21.94. \]

- The maximum volume of 3533.34 cm³ occurs when the dimensions of the box (in cm) are
  \[ 11.40 \times 11.40 \times 27.21. \]
eq =
[ y*z-L*(2*y+2*z)-4*M, x*z-L*(2*x+2*z)-4*M,
  x*y-L*(2*x+2*y)-4*M]
eq = [eq g h]
eq =
[ y*z-L*(2*y+2*z)-4*M, x*z-L*(2*x+2*z)-4*M,
  x*y-L*(2*x+2*y)-4*M, 2*x*y+2*y*z+2*x*z-1500,
  4*x+4*y+4*z-200]
c = solve(eq(1), eq(2), eq(3), eq(4), eq(5))
c =
L: [6x1 sym]
M: [6x1 sym]
x: [6x1 sym]
y: [6x1 sym]
z: [6x1 sym]
c = [c.x c.y c.z]
c =
[ 50/3+5/3*10^(1/2), 50/3+5/3*10^(1/2), 50/3-10/3*10^(1/2)]
[ 50/3-5/3*10^(1/2), 50/3-5/3*10^(1/2), 50/3-5/3*10^(1/2)]
[ 50/3+5/3*10^(1/2), 50/3-5/3*10^(1/2), 50/3-5/3*10^(1/2)]
[ 50/3-5/3*10^(1/2), 50/3+5/3*10^(1/2), 50/3-5/3*10^(1/2)]
[ 50/3-5/3*10^(1/2), 50/3+5/3*10^(1/2), 50/3-5/3*10^(1/2)]
% Crank out functions values of f.
format bank
f vals = [];
for k = 1:size(c,1)
  p = c(k,:);
  p dec = eval(p);
  func val = subs(f, [x y z], p);
  f val dec = eval(func val);
  f vals = [f vals, f val dec];
end
p dec =
  21.94 21.94 6.13
f val dec =
  2947.94
p dec =
  11.40 11.40 27.21
f val dec =
  3533.54
p dec =
  6.13 21.94 21.94
f val dec =
  2947.94
p dec =
  27.21 11.40 11.40
f val dec =
  3533.54
p dec =
  21.94 6.13 21.94
f val dec =
  2947.94
p dec =
  11.40 27.21 11.40
f val dec =
  3533.54
f vals =
Columns 1 through 5
  2947.94 3533.54 2947.94 3533.54 2947.94
Column 6
  3533.54
abs min =
  2947.94
abs max =
  3533.54