Summary

The surface area of an explicitly defined surface \( z = f(x, y) \), \((x, y) \in D\), whose first-order partial derivatives are continuous on \( D \) is given by
\[
S = \int \int_D \sqrt{1 + f_x^2 + f_y^2} \, dA.
\]
If \( D \) is a polar region, switch to polar coordinates after setting up the integral.

Hand / MATLAB Examples

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Find the surface area of the part of the plane \( 2x + 3y - z + 1 = 0 \) that lies above the rectangle \( R = [1, 4] \times [2, 4] \).

Solution

Let \( z = f(x, y) = 2x + 3y + 1 \). Then \( f_x = 2 \), \( f_y = 3 \), and the surface area is
\[
S = \int_2^4 \int_1^4 \sqrt{1 + 2^2 + 3^2} \, dx \, dy = 6\sqrt{14} \approx 22.45 \text{ cm}^2.
\]

834/4

Find the area of the part of the surface \( z = f(x, y) = x + y^2 \) that lies above the triangle with vertices \((0,0), (1,1), \) and \((0,1)\).

Solution

First, \( f_x = 1 \) and \( f_y = 2y \). Here is the region of integration.

Hence the surface area is
\[
S = \int_0^1 \int_0^y \sqrt{1 + 1^2 + (2y)^2} \, dx \, dy = \frac{1}{2} \sqrt{6} - \frac{1}{2} \sqrt{2} \approx 0.99 \text{ cm}^2.
\]
Find the surface area of the part of the circular paraboloid \( z = f(x,y) = 4 - x^2 - y^2 \) that lies above the \( xy \)-plane.

**Solution**

When the paraboloid intersects the \( xy \)-plane, we have

\[ 0 = z = 4 - x^2 - y^2. \]

Hence the curve of intersection is the circle \( x^2 + y^2 \leq 4 \), \( z = 0 \). This is the boundary of the region of integration, the circular disk \( D = \{(x,y) : x^2 + y^2 \leq 4 \} \). Thus the surface area is

\[
S = \iint_D \sqrt{1 + 4x^2 + 4y^2} \, dA = \int_0^{2\pi} \int_0^2 \sqrt{1 + 4r^2} \, r \, dr \, d\theta = \frac{\pi}{6} (17\sqrt{17} - 1) \approx 36.18 \text{ cm}^2.
\]

\[
f_x = -2x \quad \text{and} \quad f_y = -2y.
\]

Thus the surface area is

\[
S = \int_{-2}^2 \int_{-1}^1 \sqrt{1 + 4x^2 + 4y^2} \, dy \, dx = 3\sqrt{11} + \ln \left( \frac{\sqrt{11} + 3}{\sqrt{11} - 3} \right) \approx 12.94 \text{ cm}^2.
\]