Spring 2004  Math 253/501–503  
14  Vector Calculus  
14.3  The Fundamental Theorem for Line Integrals  

Tue, 06/Apr  ©2004, Art Belmonte  

**Summary**

**Framework**

Let $E \subset \mathbb{R}^n$ be a subset of $n$-dimensional space, $f : E \to \mathbb{R}$ a scalar field and, $F : E \to \mathbb{R}^n$ a vector field. Let $C$, parameterized by $g : [a, b] \to \mathbb{R}^n$, be a smooth curve ($g' \neq 0$) in $n$-D space whose range is contained in $E$, so that the compositions $f \circ g$ and $F \circ g$ are defined. Further specifications will be stated below.

**Definitions**

- For a continuous vector field $F$, the line integral $\int_C F \cdot dg$ is **independent of path** if $\int_C F \cdot dg = \int_{C_2} F \cdot dg$ for any two paths $C_1$ and $C_2$ in $E$ having the same starting points and same ending points.
- A vector field $F$ is **conservative** if $F = \nabla f$; i.e., if $F$ is the gradient of some scalar potential function $f$.
- A **closed curve** or **closed path** is one for which the starting and ending points are the same; i.e., $g(a) = g(b)$.
- The set $E$ is **open** provided that for each point $a \in E$ there is an open $n$-ball $B(a; r) = \{x \in \mathbb{R}^n : \|x - a\| < r\}$ that is wholly contained in $E$.
- A **simple** curve is one that doesn’t intersect itself (except perhaps at its endpoints).
- The set $E$ is **connected** if each pair of points in $E$ may be joined by a path that is entirely contained in $E$.
- A **simply-connected** plane region $D$ is a connected region with the additional proviso that every simple closed curve in $D$ encompasses only points that are in $D$. The idea here is that $D$ has no holes and is not separated into pieces.

**Theorems**

1. **Fundamental Theorem for Line Integrals (FTLI):** With $f$, $C$, and $g$ as specified in the framework, suppose further that $f$ is differentiable with continuous gradient $\nabla f$. We then have $\int_C \nabla f \cdot dg = f(g(b)) - f(g(a))$. [Accordingly, line integrals of conservative vector fields are path independent.]

2. The line integral $\int_C F \cdot dg$ is independent of path if and only if $\int_C F \cdot dg = 0$ for every closed path in $E$.

3. Let $F$ be continuous on an open connected region $E$ and supposed $\int_C F \cdot dg$ is independent of path in $E$. Then $F$ is conservative; i.e., $F = \nabla f$ for some scalar potential function $f$.

4. Let $F = [P, Q]$ be conservative and have continuous first order partial derivatives on a plane region $D$. Then $P_y = Q_x$ on $D$.

5. Let $F = [P, Q]$ have continuous first order partial derivatives on a simply-connected plane region $D$. Further suppose that $P_y = Q_x$ on $D$. Then $F$ is conservative; i.e., $F = \nabla f$ for some scalar potential function $f$.

**Hand Examples**

**891/2**

Determine whether $F = [P, Q] = [3x^2 - 4y, 4y^2 - 2x]$ is a conservative vector field.

**Solution**

First note that $F$ has continuous first-order partial derivatives on $D = \mathbb{R}^2$, the entire $xy$-plane. Now $P_y = -4$, whereas $Q_x = -2$. Hence $P_y \neq Q_x$. Thus $F$ is not conservative, lest Theorem 4 in the Summary be violated.

**891/9**

Determine whether $F = [P, Q] = [ye^x + \sin y, e^x + x \cos y]$ is a conservative vector field. If so, find a potential function $f$ for $F$; i.e., find $f$ such that $F = \nabla f$.

**Solution**

First note that $F$ has continuous first-order partial derivatives on $D = \mathbb{R}^2$, the entire $xy$-plane, a simply-connected region. Furthermore, $P_y = e^x + \cos y = Q_x$. Thus $F$ is conservative by Theorem 5 of the Summary.

- Construct a potential function $f$ for the vector field $F$; i.e., find $f$ such that $F = \nabla f$. Partially antidifferentiate, then harvest unique terms. (Don’t worry about constants or pseudo-constants.)

<table>
<thead>
<tr>
<th>$F:$</th>
<th>$ye^x + \sin y$</th>
<th>$e^x + x \cos y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\nabla f:$</td>
<td>$f_x$</td>
<td>$f_y$</td>
</tr>
<tr>
<td>Antidiff</td>
<td>$ye^x + x \sin y$</td>
<td>$ye^x + x \sin y$</td>
</tr>
<tr>
<td>Harvest!</td>
<td>$f(x, y) = ye^x + x \sin y$</td>
<td></td>
</tr>
</tbody>
</table>
Find a potential function \( f \) for \( \mathbf{F} = [4xe^z, \cos y, 2x^2 e^z] \), then use the FTLI to evaluate \( \int_C \mathbf{F} \cdot dg \), where the curve \( C \) is parameterized by \( \mathbf{g}(t) = [t, t^2, t^4], 0 \leq t \leq 1 \).

**Solution**

- Construct a potential function \( f \) for the vector field \( \mathbf{F} \); i.e., find \( f \) such that \( \mathbf{F} = \nabla f \).

<table>
<thead>
<tr>
<th>( \mathbf{F} )</th>
<th>( \nabla f )</th>
<th>Antidiff</th>
<th>Harvest!</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 4xe^z )</td>
<td>( f_x )</td>
<td>( 2x^2 e^z )</td>
<td></td>
</tr>
<tr>
<td>( \cos y )</td>
<td>( f_y )</td>
<td>( \sin y )</td>
<td></td>
</tr>
<tr>
<td>( 2x^2 e^z )</td>
<td>( f_z )</td>
<td>( 2x^2 e^z + \sin y )</td>
<td></td>
</tr>
</tbody>
</table>

- Now apply the FTLI.

\[
\int_C \mathbf{F} \cdot dg = \int_C \nabla f \cdot dg = f(g(1)) - f(g(0))
\]

\[
= f(1, 1, 1) - f(0, 0, 0)
\]

\[
= 2e + \sin 1 \approx 6.28
\]

**MATLAB Examples**

### 891/2 [revisited]

Determine whether \( \mathbf{F} = [P, Q] = [3x^2 - 4y, 4y^2 - 2x] \) is a conservative vector field.

**Solution**

Will that dog hunt? That is, does it have potential? If so, the **pot** command returns a potential function \( f \) for \( \mathbf{F} \). If not, it returns 0 [false]. In this case, **pot** returns 0, so that \( \mathbf{F} \) is not conservative (since it has no potential function).

### 891/9 [revisited]

Determine whether \( \mathbf{F} = [P, Q] = [ye^x + \sin y, e^x + x \cos y] \) is a conservative vector field. If so, find a potential function \( f \) for \( \mathbf{F} \); i.e., find \( f \) such that \( \mathbf{F} = \nabla f \).
Solution

We use \texttt{pot} to find a potential function, then \texttt{FTLI} to apply the Fundamental Theorem for Line Integrals.

\begin{verbatim}
% Stewart 891/20
% sym t x y unreal
v = [x y];
F = [2*y^2 - 12*x^3*y^3 4*x*y - 9*x^4*y^2];
f = pot(F,v); pretty(f)
\end{verbatim}

\begin{verbatim}
24 3
2 y x-3 x y
\end{verbatim}

\begin{verbatim}
% Manually apply the FTLI.
f = inline(char(f), 'x', 'y');
li = f(3,2) - f(1,1)
li =
-1919
\end{verbatim}

\begin{verbatim}
892/24

From a plot of the vector field \( F \) appears below. It looks as if \( F \) is conservative, since around an arbitrary simple closed path, the number and size of \( F \)'s vectors pointing in a given direction is the same as the number and size of \( F \)'s vectors pointing in the opposite direction.

By finding a potential function \( f \) for \( F \), we see that \( F \) is indeed conservative.

\begin{romanlist}
\item \begin{verbatim}
% Stewart 891/24
% sym t x y unreal
v = [x y];
F = [2*x*y + sin(y) x^2 + x*cos(y)];
f = pot(F,v); pretty(f)
\end{verbatim}

\begin{verbatim}
2
x y + x sin(y)
\end{verbatim}

\begin{verbatim}
\end{verbatim}
\end{romanlist}