Spring 2004 Math 253/501–503
14 Vector Calculus
14.5 Curl and Divergence
Thu, 08/Apr ©2004, Art Belmonte

Summary

Let \( \mathbf{F}(x, y, z) = [P, Q, R] \) be a vector field and \( f(x, y, z) \) be a scalar field.

Definitions

- The vector differential operator \( \text{del} \) is \( \nabla = \left[ \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right] \).
- The gradient of \( f \) is \( \nabla f = [f_x, f_y, f_z] \).
- The curl of \( \mathbf{F} \) is \( \text{curl} \mathbf{F} = \nabla \times \mathbf{F} = \left| \begin{array}{ccc} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{array} \right| \)
  \( = [R_y - Q_z, P_z - R_x, Q_x - P_y] \).
- The divergence of \( \mathbf{F} \) is \( \text{div} \mathbf{F} = \nabla \cdot \mathbf{F} = P_x + Q_y + R_z \).
- The laplacian of \( f \) is \( \nabla^2 f = \nabla \cdot \nabla f = f_{xx} + f_{yy} + f_{zz} \).
- The laplacian of \( \mathbf{F} \) is \( \nabla^2 P, \nabla^2 Q, \nabla^2 R \).

Theorems

- The curl of a gradient vector field is the zero vector. That is, \( \text{curl} \left( \nabla f \right) = \mathbf{0} = [0, 0, 0] \).
- If \( \mathbf{F} \) is a vector field defined on \( \mathbb{R}^3 \) such that \( \text{curl} \mathbf{F} = \mathbf{0} \) (the zero vector \([0, 0, 0] \)), then \( \mathbf{F} \) is conservative; i.e., \( \mathbf{F} = \nabla f \) for some scalar potential function \( f \).
- If \( \mathbf{F} \) is a vector field on \( \mathbb{R}^3 \) whose components have continuous second-order partial derivatives, then \( \text{div} \, \text{curl} \mathbf{F} = 0 \), the scalar zero.

Hand Examples

905/2

Find the curl & divergence of the vector field \( \mathbf{F} = [x^2 y, yz^2, x^2 z] \).

Solution

- The curl of \( \mathbf{F} \) is \( \text{curl} \mathbf{F} = \nabla \times \mathbf{F} \) or
  \[
  \begin{vmatrix}
    i & j & k \\
    \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
    x^2 & y & z^2 \\
  \end{vmatrix} = [0, 0, -\sin(x)].
  \]
- The divergence of \( \mathbf{F} \) is \( \text{div} \mathbf{F} = \nabla \cdot \mathbf{F} \) or
  \[
  \begin{vmatrix}
    \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
    x & y & z \\
  \end{vmatrix} = [x, y, z] = [0, 0, 0].
  \]

MATLAB Examples

905/6

Find the curl and divergence of \( \mathbf{F} = [\sin(x), \cos(x), z^2] \).

Solution

In MATLAB you may use the \texttt{curl} and \texttt{div} commands I wrote. (Cooper already had a different routine called \texttt{curl}.) There are also \texttt{curl} and \texttt{div} commands available on your TI-89 on the \texttt{FVMD} menu (Functions and Matrix & Vector Derivatives).

\begin{verbatim}
% Stewart 905/6

syms x y z
F = [sin(x) cos(x) z^2];
curlF = Curl(F); pretty(curlF)

divF = div(F); pretty(divF)
cos(x) + 2 z

echo off; diary off
\end{verbatim}
905/16

Is the vector field $F = [x, e^y \sin z, e^y \cos z]$ conservative? If so, find a scalar potential function $f$ for $F$.

**Solution**

We see that $\nabla \times F = \mathbf{0}$ on $\mathbb{R}^3$. So $F$ is conservative by the second theorem in the Summary. We then use the `pot` command to yield the desired potential function $f$.

```matlab
% Stewart 905/16
% sym x y z
v = [x y z];
F = [x exp(y)*sin(z) exp(y)*cos(z)];
curlF = Curl(F); pretty(curlF)
f = pot(F,v); pretty(f)

2
1/2 * x + exp(y) * sin(z)
```

906/38

Given the position vector field $r = [x, y, z]$ and spherical radius variable $\rho = \sqrt{x^2 + y^2 + z^2}$, verify the identity $\nabla^2 \rho^3 = 12 \rho$, where $\nabla^2$ is the laplacian.

**Solution**

We simply show that $\nabla^2 \rho^3 - 12 \rho = 0$. Done!

```matlab
% Stewart 906/38
% sym x y z
v = [x y z];
r = len(v)
r = (x^2 + y^2 + z^2)^(1/2)
f = r^3;
% Left - Right = 0
lmrez = simple(laplacian(f) - 12*r)
lmrez =
0
```

% echo off; diary off