**Surface Integrals**

**Applications of Surface Integrals**

Let $\delta$ be the mass density and $\sigma$ the charge density. The center of mass is also called the centroid when the density is constant.

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<th>Application</th>
<th>Surface Integrals</th>
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<td>scalar differential</td>
<td>$dS = |s_u \times s_v| \ du \ dv$</td>
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<tr>
<td>measure</td>
<td>$A(S) = \iint_S 1 \ dS$</td>
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<tr>
<td>total mass</td>
<td>$m = \iint_S \delta \ dS$</td>
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<tr>
<td>electric charge</td>
<td>$Q = \iint_S \sigma \ dS$</td>
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<tr>
<td>moments</td>
<td>$M_{yz} = \iint_S x \delta \ dS$</td>
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<tr>
<td>center of mass</td>
<td>$\bar{x} = \frac{1}{m} \int_S \bar{x} \ dS$</td>
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<tr>
<td>moments of inertia</td>
<td>$I_x = \iint_S (y^2 + z^2) \delta \ dS$</td>
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<tr>
<td>radii of gyration</td>
<td>$\bar{x} = \sqrt{I_x/m}, \ \bar{y} = \sqrt{I_y/m}, \ \bar{z} = \sqrt{I_z/m}$</td>
</tr>
</tbody>
</table>

**Hand / MATLAB Examples**

925/4

Evaluate the surface integral $\iint_S xz \ dS$, where the surface is the triangular patch with vertices $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$.

**Solution**

- The surface is pictured at left above. Its projection $D$ onto the $xy$-plane appears to the right of it.
- The routine `plane3pt` on your TI-89 or in MATLAB gives $x + y + z = 1$ or $z = 1 - x - y$ as an equation of the plane through the three noncollinear vertices.
- A rectangular parameterization of the surface is $s(x, y) = [x, y, 1 - x - y], \ 0 \leq y \leq 1 - x, \ 0 \leq x \leq 1$.
- Compute $s_x \times s_y = [1, 0, -1] \times [0, 1, -1] = [1, 1, 1]$. Then $\|s_x \times s_y\| = \sqrt{3}$. With $f(x, y, z) = xz$, we have

  $\iint_S f \ dS = \iint_D f(s(x, y)) \|s_x \times s_y\| \ dy \ dx$

  $= \int_0^1 \int_0^{1-x} x(1 - x - y)\sqrt{3} \ dy \ dx$

  $= \frac{1}{24}\sqrt{3} \approx 0.0722$.

925/4 [revisited]

Evaluate the surface integral $\iint_S xz \ dS$, where the surface is the triangular patch with vertices $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$.  

**Table Notes**

1. Other first-order moments are symmetrically defined.
   
   $M_{xz} = \iint_S y \delta \ dS, \ \ M_{xy} = \iint_S z \delta \ dS$

2. Other second-order moments are symmetrically defined.
   
   $I_y = \iint_S (x^2 + z^2) \delta \ dS, \ \ I_z = \iint_S (x^2 + y^2) \delta \ dS$
Solution

Clearly there is a lot of work involved in computing a surface integral by hand. (It is a nice review of all the material in the course, however!) Recall from Section 14.6 that the sis routine (available on the TI-89 and in MATLAB) computes the exact value of the surface integral of a scalar field automatically. Here’s the needful. (If the surface integral of a scalar field cannot be computed exactly, just use sisn in MATLAB.)

% Stewart 925/4
%syms x y z
%f = x*z;
%s = [x y 1-x-y];
%M = [y 0 1-x; x 0 1];
%S = sis(f,s,M);
pretty(S)

floated = double(S)
floated = 0.0722
% echo off; diary off

925/6

Evaluate the surface integral \( \iint_S y^2 + z^2 \, dS \), where the surface is the part of the circular paraboloid \( x = 4 - y^2 - z^2 \) that lies in front of the plane \( x = 0 \), the \( yz \)-plane.

Solution

This time we’ll just use sis.

% Stewart 925/6
%syms r t x y z
%f = y^2 + z^2;
%s = [4-r^2 r*cos(t) r*sin(t)];
%M = [r 0 2; t 0 2*pi];
%S = sis(f,s,M);
pretty(S)

floated = double(S)
floated = 84.4635
% echo off; diary off

925/12

Evaluate the surface integral \( \iint_S x^2 + y^2 + z^2 \, dS \), where the surface consists of the part of the circular cylinder \( x^2 + y^2 = 9 \) between the planes \( z = 0 \) and \( z = 2 \) together with its top and bottom circular disks; in other words, a closed tin can!

Solution

OK, campers: we have three surface integrals to compute. Let’s set up parameterizations for the side, bottom, and top of the tin can, then dispatch the whole shooting match with repeated invocations of sis added together!
Evaluate the surface integral \( \int_C x^2 + y^2 \, dS \), where the surface is the helicoid (spiral ramp)

\[ s(u, v) = [u \cos v, u \sin v, v] \cdot 0 \leq u \leq 1, \ 0 \leq v \leq \pi. \]

**Solution**

Again we use \( \text{sis} \).

\[
S = \frac{4}{3} \pi \\
\text{pretty}(S) \\
\text{floated} = \text{double}(S) \\
floated = 4.1888 \\
\% \ echo \ off; \ diary \ off
\]

---

**925/16**

Evaluate the surface integral \( \iint_S \mathbf{w} \cdot dS \) of the vector field \( \mathbf{w} = [x^2 y, -3x y^2, 4y^3] \) over the part of the circular paraboloid \( z = x^2 + y^2 - 9 \) that lies below the rectangle \( 0 \leq x \leq 2, \ 0 \leq y \leq 1 \) and has downward orientation. That is, compute the flux of \( \mathbf{w} \) across this surface.

**Solution**

- The surface appears above.
- You are handed a rectangular parameterization of the surface:
  \[ s(x, y) = [x, y, x^2 + y^2 - 9], \ 0 \leq x \leq 2, \ 0 \leq y \leq 1. \]
- Compute \( \mathbf{s}_x \times \mathbf{s}_y = [1, 0, 2x] \times [0, 1, 2y] = [-2x, -2y, 1] \), an upward orientation. (Look at its \( k \) component!) Since the author requested a downward orientation, we'll negate the double integral we compute below.

\[
- \iint_S \mathbf{w} \cdot dS = - \iint_D \mathbf{w}(s(x, y)) \cdot (\mathbf{s}_x \times \mathbf{s}_y) \, dx \, dy \\
= - \int_0^1 \int_0^2 \left[ x^2 y, -3x y^2, 4y^3 \right] \cdot [-2x, -2y, 1] \, dx \, dy \\
= - \int_0^1 \int_0^2 4y^3 + 6xy^3 - 2x^3 y \, dx \, dy \\
= -1.
\]

**925/16 [revisited]**

Evaluate the surface integral \( \iint_S \mathbf{w} \cdot dS \) of the vector field \( \mathbf{w} = [x^2 y, -3x y^2, 4y^3] \) over the part of the circular paraboloid
Solution

This time we invoke \texttt{siv}, which computes the exact value of the surface integral of a vector field over a surface. We still need to check the orientation, but MATLAB sure beats doing it by hand! NOTE WELL: Internally, \texttt{siv} computes the orientation as \texttt{s u \times s v}, where \texttt{u} is the inner variable in the range matrix \texttt{M} and \texttt{v} is the outer variable. Accordingly, you should use this same order when computing orientation to decide whether or not to negate the result returned by \texttt{siv}.

\begin{verbatim}
\% Stewart 925/16
\% sym x y z
\w = [x^2*y -3*x*y^2 4*y^3];
\s = [x y x^2+y^2-9];
\M = [x 0; y 1];
\%
\% For our parameterization, the normal points UPWARD. The author wants DOWNWARD.
\sx = diff(s,x); sy = diff(s,y); c = cross(sx,sy)
c = [-2*x, -2*y, 1]
\%
\% Accordingly, negate the result that siv returns!
\S = -siv(w,s,M)
\S = -1
\pretty(S)
\floated = double(S)
floated = -1
\%
\end{verbatim}

\textbf{Solution}

Here we just use \texttt{siv}. The flux is $\frac{73}{6}\pi \approx 38.22$.

\begin{verbatim}
\% Stewart 925/18
\% sym r t x y z
\w = [-x -y z^2];
\s = [r*cos(t); r*sin(t); r];
\M = [r 1; t 0 2*pi];
\%
\% For our parameterization, the normal points UPWARD, as the author has requested.
\sr = diff(s,r); st = diff(s,t); c = simple(cross(sr,st))
c = [-r*cos(t), -r*sin(t), r]
% Accordingly, use the result that siv returns!
\S = siv(w,s,M)
\S = 73/6*pi
\pretty(S)
\floated = double(S)
floated = 38.2227
\%
\end{verbatim}

\textbf{Solution}

Find the mass (in kg) of a thin funnel in the shape of a cone $z = x^2 + y^2$, $1 \leq z \leq 4$, if its variable density is $\delta = 10 - z$.

\begin{verbatim}
\% Stewart 926/32
\% sym r t x y z
\f = 10 - z;
\s = [r*cos(t); r*sin(t); r];
\M = [r 1; t 0 2*pi];
\%
\end{verbatim}

\textbf{Solution}

The mass is $\int_S \delta \, dS = 108\sqrt{2}\pi \approx 479.83$ kg, the surface integral of a scalar field, dispatched with \texttt{sis}.
A fluid has density $\delta = 1500$ and velocity $\mathbf{v} = [-y, x, 2z]$. Find the rate of flow outward through the sphere $x^2 + y^2 + z^2 = 25$.

**Solution**

That hand work we just did was a world ‘o’ hurt! Instead, let’s use `siv` to compute the surface integral of the vector field $\mathbf{w} = \delta \mathbf{v}$ over the sphere. Herewith the needful in MATLAB. Much better!

```matlab
\% Stewart 926/36
\% syms p t x y z
v = [-y x 2*z];
w = 1500*v;
s = [5*sin(p)*cos(t) 5*sin(p)*sin(t) 5*cos(p)];
M = [p 0 pi; t 0 2*pi];
sp = diff(s,p); st = diff(s,t);
c = simple(cross(sp,st))
c = [-25*(1+cos(p)^2)*cos(t), -25*(-1+cos(p)^2)*sin(t),
     25*cos(p)*sin(p)]
S = siv(w,s,M)
S =
500000*pi
pretty(S)
floated = double(S)
floated =
500000 pi
```

926/36

A fluid has density $\delta = 1500$ and velocity $\mathbf{v} = [-y, x, 2z]$. Find the rate of flow outward through the sphere $x^2 + y^2 + z^2 = 25$.

**Solution**

That hand work we just did was a world ‘o’ hurt! Instead, let’s use `siv` to compute the surface integral of the vector field $\mathbf{w} = \delta \mathbf{v}$ over the sphere. Herewith the needful in MATLAB. Much better!

```matlab
\% Stewart 926/36
\% syms p t x y z
v = [-y x 2*z];
w = 1500*v;
s = [5*sin(p)*cos(t) 5*sin(p)*sin(t) 5*cos(p)];
M = [p 0 pi; t 0 2*pi];
sp = diff(s,p); st = diff(s,t);
c = simple(cross(sp,st))
c = [-25*(-1+cos(p)^2)*cos(t), -25*(-1+cos(p)^2)*sin(t),
     25*cos(p)*sin(p)]
S = siv(w,s,M)
S =
500000*pi
pretty(S)
floated = double(S)
floated =
500000 pi
```