Fall 2007 Math 253
Exam 3A: Solutions
Mon, 05/Nov ©2007 Art Belmonte

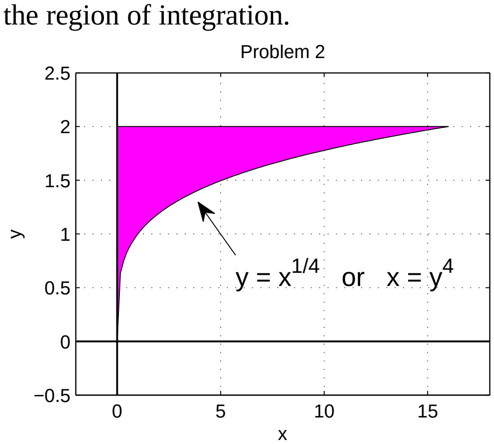
1. The volume is $\iint_D f(x,y)\,dA$, which we'll estimate. Recall that $f(x,y) = \sqrt{52 - x^2 - y^2}$.

   - While you are more than welcome to use the relevant TAMUCALC approximate integration facility (with $m = n = 2$), here is how things are done by hand.
   - Now $\Delta x = h = \dfrac{b-a}{m} = \dfrac{4-2}{2} = 1$ and $\Delta y = k = \dfrac{d-c}{n} = \dfrac{6-2}{2} = 2$.
   - Let $\left(x_{ij}^*, y_{ij}^*\right)$ represent the center of the ij^{th}-subrectangle of the partition. The double Riemann sum is given by

$$\begin{aligned} & \sum_{i=1}^{m}\sum_{j=1}^{n} f\left(x_{ij}^*, y_{ij}^*\right)\Delta x_i\,\Delta y_j \\ = & \sum_{i=1}^{2}\sum_{j=1}^{2} f\left(x_{ij}^*, y_{ij}^*\right)\Delta x\,\Delta y \\ = & \; 2\sum_{i=1}^{2}\sum_{j=1}^{2} f\left(x_{ij}^*, y_{ij}^*\right) \end{aligned}$$

   since $\Delta x = 1$ and $\Delta y = 2$. This is twice the sum of the function values at the centers. This value is $2\left(f\left(\frac{5}{2},3\right) + f\left(\frac{7}{2},3\right) + f\left(\frac{5}{2},5\right) + f\left(\frac{7}{2},5\right)\right)$ or $\sqrt{83} + \sqrt{59} + \left(\sqrt{41} + 7\right)\sqrt{3} \approx 40.01\ \text{cm}^3$.

2. To evaluate $\displaystyle\int_0^{16}\int_{\sqrt[4]{x}}^{2} \frac{1}{y^5+1}\,dy\,dx$ by reversing the order of integration, we start by drawing a picture of the region of integration.



Reverse the order of integration and proceed.

$$\begin{aligned} & \int_0^2\int_0^{y^4} \frac{1}{y^5+1}\,dx\,dy \\ = & \int_0^2 \left.\frac{x}{y^5+1}\right|_{x=0}^{x=y^4} dy \\ & \int_0^2 \frac{y^4}{y^5+1}\,dy \\ = & \left.\tfrac{1}{5}\ln\left(y^5+1\right)\right|_0^2 \\ = & \frac{\ln 33}{5} - \frac{\ln 1}{5} = \frac{\ln 33}{5} \approx 0.70. \end{aligned}$$

3. Compute the mass of the plate, then its center of mass.

   - The mass is $m = \iint_D \delta\,dA = \int_0^{\pi}\int_0^{\sin x} xy\,dy\,dx = \dfrac{\pi^2}{8} = 1.23.$
   - The center of mass is

$$\begin{aligned} [\bar{x}, \bar{y}] &= \frac{1}{m}\iint_D \delta\,[x,y]\,dA \\ &= \frac{1}{\pi^2/8}\int_0^{\pi}\int_0^{\sin x} xy\,[x,y]\,dy\,dx \\ &= \left[\frac{2\pi^2-3}{3\pi}, \frac{16}{9\pi}\right] \approx [1.78, 0.57]\,. \end{aligned}$$

4. Compute the mass of the solid, then its center of mass.

   - Via **cline2pt**, the slanted line in the shadow region is $y = 2 - 2x$.
   - Via **plane3pt**, the upper face of the solid lies in the plane $z = 3 - 3x - \frac{3}{2}y$.
   - The mass is $m = \iiint_E \delta\,dV$ or $\displaystyle\int_0^1\int_0^{2-2x}\int_0^{3-3x-\frac{3}{2}y} x^2+y^2+z^2\,dz\,dy\,dx = \frac{7}{5} = 1.40.$
   - The center of mass is

$$\begin{aligned} & [\bar{x}, \bar{y}, \bar{z}] \\ = & \frac{1}{m}\iiint_E \delta\,[x,y,z]\,dA \\ = & \int_0^1\int_0^{2-2x}\int_0^{3-3x-\frac{3}{2}y} \left(x^2+y^2+z^2\right)[x,y,z]\,dz\,dy\,dx \\ = & \left[\frac{4}{21}, \frac{11}{21}, \frac{8}{7}\right] \approx [0.19, 0.52, 1.14]\,. \end{aligned}$$

[See the next page for the rest of the problems.]

5. The part of the hyperbolic paraboloid $z = y^2 - x^2$ in question lies over a ring D in the xy-plane.

   - Now $f_x = -2x$ and $f_y = 2y$. Accordingly, the surface area is

$$\begin{aligned} S &= \iint_D \sqrt{1 + f_x^2 + f_y^2}\, dA \\ &= \iint_D \sqrt{1 + 4x^2 + 4y^2}\, dA \\ &= \int_0^{2\pi} \int_3^5 \sqrt{1 + 4r^2}\, r\, dr\, d\theta \\ &= \tfrac{\pi}{6} \left(101\sqrt{101} - 37\sqrt{37}\right) \approx 413.63 \text{ cm}^2. \end{aligned}$$

6. In cylindrical coordinates, an equation of the cone $z = \sqrt{x^2 + y^2}$ is $z = r$. This is $\rho \cos\phi = \rho \sin\phi$ in spherical coordinates. Thus $\tan\phi = 1$ or $\phi \equiv \frac{\pi}{4}$ on the cone. This gives us the *lower* limit for ϕ in the triple integral below. The upper limit is $\phi = \frac{\pi}{2}$ when we hit the xy-plane. Therefore, the volume is

$$\begin{aligned} V &= \iiint_E 1\, dV \\ &= \int_0^{2\pi} \int_{\pi/4}^{\pi/2} \int_0^2 1\, \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta \\ &= \tfrac{8}{3}\sqrt{2}\pi \approx 11.85 \text{ cm}^3. \end{aligned}$$

   (This is the direct way to do the problem. There are other ways. For example, you may subtract a volume from the volume of a hemisphere and/or use cylindrical coordinates.)

7. This problem involves a change of variables in a multiple integral.

   - The lines that form the boundary of the region of integration are

$$x + y = 2 \quad x + y = 4 \quad x - y = 6 \quad x - y = 8.$$

     Using the inverse transformation

$$T^{-1} :\ u = \frac{x + y}{2},\ v = \frac{x - y}{2}$$

     we see that these correspond to the following lines in the uv-plane:

$$u = 1 \quad u = 2 \quad v = 3 \quad v = 4.$$

   - Recall the forward transformation

$$T :\ x = u + v,\ y = u - v.$$

     The Jacobian matrix is $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, the Jacobian determinant is $-1 - 1 = -2$, and the Jacobian factor is $J = 2$.

   - We may now transform the integral

$$\iint_R e^{(x+y)/2} \ln\left(\frac{x - y}{2}\right) dx\, dy$$

     and dispatch it.

$$\begin{aligned} &\iint_R f(x, y)\, dx\, dy \\ =\ &\iint_{D_{uv}} f(x(u, v), y(u, v))\, J\, du\, dv \\ =\ &\int_1^2 \int_3^4 2e^u \ln v\, dv\, du \\ =\ &-2(3\ln 3 - 8\ln 2 + 1)\, e\, (e - 1) \\ \approx\ &11.67 \end{aligned}$$