Spring 2004 Math 253/501–503
13 Multiple Integrals
13.11 Change of Variables in Multiple Integrals
Tue, 23/Mar ©2004, Art Belmonte

Summary

Framework

Let $\mathbb{R}^n_{\mathbf{w}}$ and $\mathbb{R}^n_{\mathbf{x}}$ be copies of n-dimensional space. (For example, when $n = 2$, we might speak of the $w_1 w_2$-plane and $x_1 x_2$-plane, or the uv-plane and xy-plane.) Suppose that $Q \subset \mathbb{R}^n_{\mathbf{x}}$ and $S \subset \mathbb{R}^n_{\mathbf{w}}$ are measurable subsets of $\mathbb{R}^n$. (That is, for $n = 1$, the subsets have length; for $n = 2$, they have area; for $n = 3$, volume; etc.) Now let $T : \mathbb{R}^n_{\mathbf{w}} \to \mathbb{R}^n_{\mathbf{x}}$ be a continuously differentiable one-to-one transformation whose inverse transformation T^{-1} is also continuously differentiable. Suppose that $T(S) = Q$; i.e., T maps S onto Q in a one-to-one fashion.

Define the square $n \times n$ **Jacobian matrix J**, a matrix of first-order partial derivatives, as follows.

$$\mathbf{J} = \begin{bmatrix} \partial x_1/\partial w_1 & \partial x_1/\partial w_2 & \cdots & \partial x_1/\partial w_n \\ \partial x_2/\partial w_1 & \partial x_2/\partial w_2 & \cdots & \partial x_2/\partial w_n \\ \vdots & \vdots & \ddots & \vdots \\ \partial x_n/\partial w_1 & \partial x_n/\partial w_2 & \cdots & \partial x_n/\partial w_n \end{bmatrix}$$

The **Jacobian [determinant]** is the determinant of the Jacobian matrix.

Finally, let f be continuous on Q.

Change of Variables Formula for Multiple Integrals

$$\int_Q f(\mathbf{x})\, d\mathbf{x} = \int_S f(T(\mathbf{w}))\, |\det(\mathbf{J})|\; d\mathbf{w}$$

(These are n-fold integrals; e.g., triple integrals when $n = 3$.)

Hand / MATLAB Examples

The summary described the general case for change of variables. In these examples, we'll examine typical cases: when $n = 2$ (double integrals) or $n = 3$ (triple integrals).

Example A: Polar coordinates revisited

Here $x = r\cos\theta$ and $y = r\sin\theta$. Therefore,

$$\mathbf{J} = \begin{bmatrix} x_r & x_\theta \\ y_r & y_\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}.$$

Thus $|\det \mathbf{J}| = \left| r\cos^2\theta + r\sin^2\theta \right| = |r| = r$, under the proviso that $r \ge 0$, which is conventional for polar coordinates. Hence

$$\iint_{D_{xy}} f(x, y)\, dx\, dy = \iint_{D_{r\theta}} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta,$$

just like we saw in Section 13.5!

Example B: Cylindrical coordinates revisited

Here $x = r\cos\theta$, $y = r\sin\theta$, and $z = z$. Therefore,

$$\mathbf{J} = \begin{bmatrix} x_r & x_\theta & x_z \\ y_r & y_\theta & y_z \\ z_r & z_\theta & z_z \end{bmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus $|\det \mathbf{J}| = \left| 1^{3+3}\left(r\cos^2\theta + r\sin^2\theta \right) \right| = |r| = r$, under the proviso that $r \ge 0$ for cylindrical coordinates. Hence

$$\iiint_{D_{xyz}} f(x, y, z)\, dx\, dy\, dz = \iiint_{D_{r\theta z}} f(r\cos\theta, r\sin\theta, z)\, r\, dz\, dr\, d\theta,$$

just like we saw in Section 13.10!

Example C: Spherical coordinates revisited

Here $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, and $z = \rho\cos\phi$. As we see in the MATLAB diary file below, $|\det \mathbf{J}| = \rho^2\sin\phi$. Thus

$$\begin{aligned} &\iiint_{D_{xyz}} f(x, y, z)\, dx\, dy\, dz \\ = \; &\iiint_{D_{\rho\phi\theta}} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi) \cdot \rho^2\sin\phi\, d\rho\, d\phi\, d\theta, \end{aligned}$$

just like we saw in Section 13.10!

```
%
% Jacobian factor for spherical coordinates
%
syms phi rho theta
T = [rho*sin(phi)*cos(theta) ...
        rho*sin(phi)*sin(theta) ...
        rho*cos(phi)];
v = [rho phi theta]

v =

[   rho,   phi, theta]

J = jacobian(T,v)

J =

[      sin(phi)*cos(theta),  rho*cos(phi)*cos(theta), -rho*sin(phi)*sin(theta)]
[      sin(phi)*sin(theta),  rho*cos(phi)*sin(theta),  rho*sin(phi)*cos(theta)]
[                cos(phi),            -rho*sin(phi),                        0]

J_det = simple(det(J));
pretty(J_det)
                                                2
                                   sin(phi) rho
%
echo off; diary off
```

861/5

Find the Jacobian matrix and determinant of the transformation

$$T: \quad x = u + v + w, \quad y = u + v - w, \quad z = u - v + w.$$

Solution

The Jacobian matrix is $\mathbf{J} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$. Its determinant is

$$1(1 - 1) - 1(1 + 1) + 1(-1 - 1) = -4.$$

862/10

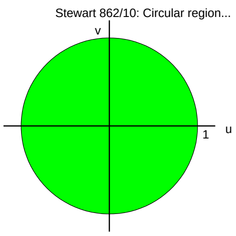
Find the image of the circular disk $S = \left\{(u, v) : u^2 + v^2 \le 1\right\}$ under the transformation $T : x = au, y = bv$, where a and b are positive constants.

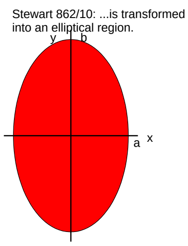
Solution

Note that $u = x/a$ and $v = y/b$. Hence the image

$$T(S) = Q = \left\{(x, y) : \frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1\right\}$$

is an elliptical region.



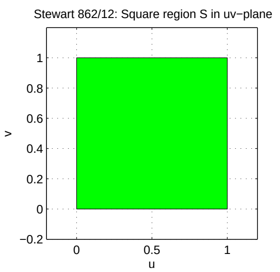


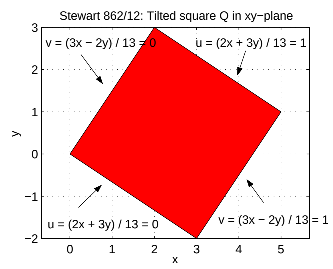
862/12

Use the transformation $T : x = 2u + 3v, y = 3u - 2v$ to evaluate the $\iint_Q x + y\, dA$. Here Q is a tilted square in the xy-plane with vertices $(0, 0)$, $(3, -2)$, $(5, 1)$, and $(2, 3)$.

Solution

The inverse image $T^{-1}(Q) = S$ is the square at left below.





The Jacobian matrix, $\mathbf{J} = \begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 3 & -2 \end{bmatrix}$, has determinant $-4 - 9 = -13$, the absolute value of which is 13. So $\iint_Q x + y\, dx\, dy = \iint_S (x(u, v) + y(u, v))\, |\det(\mathbf{J})|\ du\, dv$.

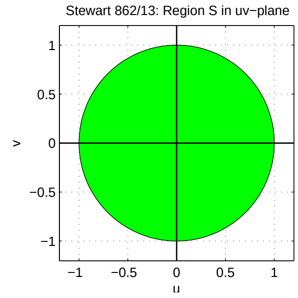
$$\int_0^1 \int_0^1 (5u + v) \cdot 13\, du\, dv = 39.$$

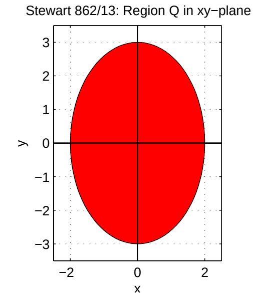
862/13

Use the transformation $T : x = 2u, y = 3v$ to evaluate the $\iint_Q x^2\, dA$. Here Q is the elliptical region $9x^2 + 4y^2 \le 36$.

Solution

Substitute $x = 2u, y = 3v$ into $9x^2 + 4y^2 \le 36$, then divide by 36 to obatin $u^2 + v^2 \le 1$. Hence the inverse image $T^{-1}(Q) = S$ is the circular region depicted at left below. (This transformation is a special case of that used in 862/10.)





The Jacobian matrix, $\mathbf{J} = \begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$, has determinant and absolute value 6. Therefore,

$$\begin{aligned}
\iint_Q x^2\, dx\, dy &= \iint_S x^2(u, v)\, |\det(\mathbf{J})|\ du\, dv \\
&= \iint_S 4u^2 \cdot 6\, du\, dv \\
&= \int_0^{2\pi} \int_0^1 24\, (r \cos\theta)^2 \cdot r\, dr\, d\theta \\
&= 6\pi \approx 18.85,
\end{aligned}$$

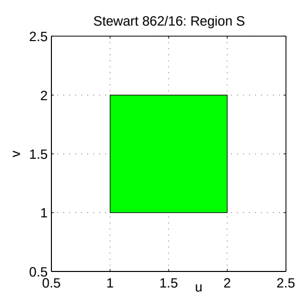
where we switched to polar coordinates in the last stage; that is, $u = r\cos\theta, v = r\sin\theta$.

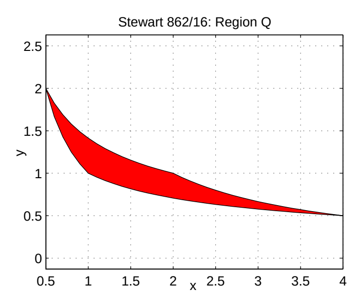
862/16

Use the transformation T whose inverse transformation is given by $T^{-1} : u = xy, v = xy^2$ to evaluate the integral $\iint_Q y^2\, dx\, dy$, where Q is the region bounded by the curves $xy = 1$, $xy = 2$, $xy^2 = 1$, and $xy^2 = 2$.

Solution

The inverse image $T^{-1}(Q) = S$ is the square at left below.





Solving $u = xy, v = xy^2$ for x and y yields $x = u^2/v, y = v/u$. Thus $\mathbf{J} = \begin{bmatrix} \frac{2u}{v} & -\frac{u^2}{v^2} \\ -\frac{v}{u^2} & \frac{1}{u} \end{bmatrix}$, the determinant of which is $1/v$. The absolute value of said determinant is $1/v$ (since $1 \le v \le 2$).

Therefore,

$$\begin{aligned}\iint_Q y^2\,dx\,dy &= \iint_S y^2(u,v)\,|\det(\mathbf{J})|\;du\,dv\\ &= \int_1^2\int_1^2 (v/u)^2\cdot\frac{1}{v}\,du\,dv\\ &= \frac{3}{4} = 0.75.\end{aligned}$$

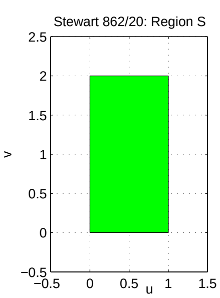
```
%
% Stewart 862/16
%
syms u v
w = [u v];
T = [u^2/v v/u];
J = jacobian(T,w)

J =

[    2*u/v, -u^2/v^2]
[   -v/u^2,      1/u]

JF = det(J)

JF =

1/v

value = int(int((v/u)^2 * JF, u,1,2), v,1,2)

value =

3/4

floated = eval(value)
floated =
    0.7500
%
echo off; diary off
```

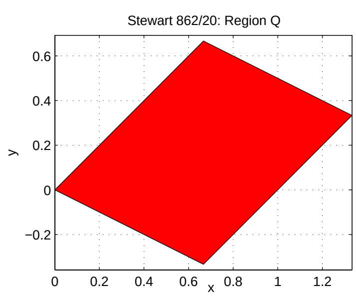
862/20

Evaluate $\iint_Q \dfrac{x+2y}{\cos(x-y)}\,dx\,dy$, where Q is the region bounded by a parallelogram whose boundaries are the straight lines $y=x$, $y=x-1$, $x+2y=0$, and $x+2y=2$.

Solution

Let $T^{-1}: u = x-y, v = x+2y$, an inverse transformation suggested by the integrand as well as the region. The boundaries of the region $T^{-1}(Q) = S$ in the uv-plane are $u=0$, $u=1$, $v=0$, and $v=2$, a rectangular region.





The forward transformation (solving for x and y) is

$$T: x = \frac{2}{3}u + \frac{1}{3}v,\ y = -\frac{1}{3}u + \frac{1}{3}v.$$

Hence the Jacobian is $\mathbf{J} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3}\\ -\frac{1}{3} & \frac{1}{3}\end{bmatrix}$. Its determinant is $\frac{1}{3}$, the absolute value of which is $\frac{1}{3}$. Accordingly,

$$\begin{aligned}\iint_Q \frac{x+2y}{\cos(x-y)}\,dx\,dy &= \iint_S \frac{v}{\cos u}\,|\det(\mathbf{J})|\;du\,dv\\ &= \int_0^1\int_0^2 \frac{v}{\cos u}\cdot\frac{1}{3}\,dv\,du\\ &= \frac{2}{3}\ln\left(\frac{1+\sin 1}{\cos 1}\right) = 0.82.\end{aligned}$$

```
%
% Stewart 862/20
%
syms u v
w = [u v];
T = [2/3*u + 1/3*v; -1/3*u + 1/3*v];
J = jacobian(T,w)

J =

[  2/3,  1/3]
[ -1/3,  1/3]

JF = det(J)

JF =

1/3

value = simple( int(int(v/cos(u) * JF, v,0,2), u,0,1) );
pretty(value)

                         2/3 log(1 + sin(1)) - 2/3 log(cos(1))
floated = eval(value)
floated =
    0.8175
%
echo off; diary off
```

Example A

Use the transformation

$$T:\quad x = \frac{r}{w}\cos\theta,\quad y = \frac{r}{w}\sin\theta,\quad z = r^2$$

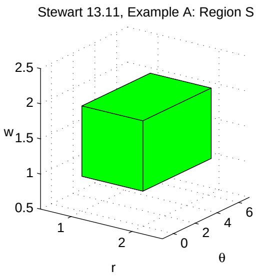
to compute the volume of the solid Q that lies between the two paraboloids $z = x^2+y^2$, $z = 4\left(x^2+y^2\right)$, and the two planes $z=1$, $z=4$.

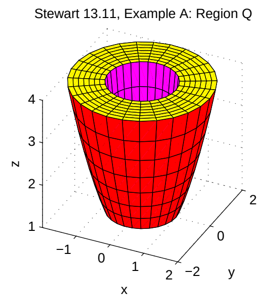
Solution

The parabolid $z = x^2+y^2$ corresponds to the case where $w=1$, whereas $z = 4\left(x^2+y^2\right)$ corresponds to $w=2$. As w varies between 1 and 2, a family of paraboloids sweeps out the solid. Accordingly the inverse image $T^{-1}(Q) = S$ is just a rectangular box in $r\theta w$-space:

$$1\le r\le 2,\quad 0\le\theta\le 2\pi,\quad 1\le w\le 2.$$

In the diary file below, we compute the needful. The volume is seen to be $\frac{45}{8}\pi \approx 17.67\ \text{cm}^3$.






```
%
% Stewart 13.11/Example A
%
syms r t w
T = [r/w*cos(t)  r/w*sin(t)  r^2];
v = [r t w];
J = jacobian(T,v)

J =

[    1/w*cos(t),   -r/w*sin(t), -r/w^2*cos(t)]
[    1/w*sin(t),    r/w*cos(t), -r/w^2*sin(t)]
[           2*r,             0,             0]

JF = simple(det(J)); pretty(JF)

                                              3
                                             r
                                          2 ----
                                              3
                                             w
value = int(int(int(1 * JF, r,1,2), t,0,2*pi), w,1,2);
pretty(value)

                                          45/8 pi

floated = eval(value)
floated =
   17.6715
%
echo off; diary off
```

[You can verify the volume of this solid of revolution by using the method of washers from Calc 2. Try it!]

864/76

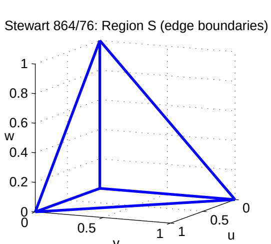
Use the transformation

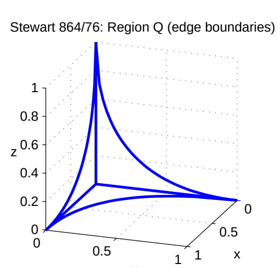
$$T: \quad x = u^2, \quad y = v^2, \quad z = w^2$$

to compute the volume of the solid Q in the first octant that is bounded by the surface $\sqrt{x} + \sqrt{y} + \sqrt{z} = 1$ and the three coordinate planes.

Solution

The inverse image is $T^{-1}(Q) = S$ is a tetrahedron in uvw-space with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$. In the diary file below, we see that the volume is $\frac{1}{90} \approx 0.0111\ \mathrm{cm}^3$.






```
%
% Stewart 864/76
%
syms u v w
T = [u^2 v^2 w^2]

T =

[ u^2, v^2, w^2]

V = [u v w];
J = jacobian(T,V)

J =

[ 2*u,   0,   0]
[   0, 2*v,   0]
[   0,   0, 2*w]

JF = det(J)

JF =

8*u*v*w

value = int(int(int(1 * JF, w,0,1-u-v), v,0,1-u), u,0,1);
pretty(value)

                                     1/90

floated = eval(value)
floated =
    0.0111
%
echo off; diary off
```