

Spring 2004 Math 253/501–503

13 Multiple Integrals

13.2 Iterated Integrals

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Summary

The double Riemann sums of §13.1 are useful for approximations when antidifferentiation is impossible. Yet computing the exact value of a double integral by taking a limit of double Riemann sums is very difficult even when it is possible.

A practical way to evaluate multiple integrals is via iterated integration, where we compute single integrals in succession. In other words, we repeatedly compute antiderivatives and apply the Fundamental Theorem of Calculus, working from inside-out until we are finished.

This mechanistic approach has been fully automated, as in the **smi** (stepwise [multiple] integration) commands on the TI-89 and in MATLAB. Accordingly, it reduces the problem of computing multiple integrals to simply setting them up. That said, you ought to try a few homework problems purely by hand to get some feeling for the work that you are being spared.

Fubini's Theorem

If f is continuous on a rectangular region $R = [a, b] \times [c, d]$, then
$$\iint_R f(x, y) dA = \int_a^b \int_c^d f(x, y) dy dx = \int_c^d \int_a^b f(x, y) dx dy.$$
 (The theorem is actually true if f is bounded on R , discontinuous on at most a finite number of smooth curves, and the iterated integrals exist.)

Hand Examples

Aids to our hand work are the TAMUCALC **Muint** menu and the **smi** command. The former contains a variety of multiple integral templates that facilitate typing. The latter shows the steps involved in integration.

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Calculate the iterated integral
$$\int_0^4 \int_0^2 x\sqrt{y} dx dy.$$

Solution

Here are the steps involved in the multiple integration. Both an exact answer and an approximation are provided.

$$\begin{aligned} \int_0^4 \int_0^2 x\sqrt{y} dx dy &= \int_0^4 \frac{1}{2}x^2y^{1/2} \Big|_{x=0}^{x=2} dy \\ &= \int_0^4 2y^{1/2} - 0 dy \\ &= \frac{4}{3}y^{3/2} \Big|_{y=0}^{y=4} \\ &= \frac{32}{3} - 0 = \frac{32}{3} = 10\frac{2}{3} \approx 10.67 \end{aligned}$$

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Calculate the iterated integral
$$\int_0^2 \int_0^3 e^{x-y} dy dx.$$

Solution

Here are the steps involved in the multiple integration. Both an exact answer and an approximation are provided.

$$\begin{aligned} \int_0^2 \int_0^3 e^{x-y} dy dx &= \int_0^2 (-e^{x-y}) \Big|_{y=0}^{y=3} dx \\ &= \int_0^2 -e^{x-3} + e^x dx \\ &= (-e^{x-3} + e^x) \Big|_{x=0}^{x=2} \\ &= (e^2 - e^{-1}) - (1 - e^{-3}) \\ &= e^2 - e^{-1} + e^{-3} - 1 \approx 6.07 \end{aligned}$$

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Calculate the double integral
$$\iint_R \frac{1+x}{1+y} dA$$
 over the rectangular region $R = [-1, 2] \times [0, 1]$.

Solution

Via the Fubini's Theorem, we have

$$\begin{aligned} \iint_R \frac{1+x}{1+y} dA &= \int_{-1}^2 \int_0^1 \frac{1+x}{1+y} dy dx \\ &= \int_{-1}^2 (1+x) \ln(1+y) \Big|_{y=0}^{y=1} dx \\ &= \int_{-1}^2 (1+x) \ln 2 - 0 dx \\ &= \left(x + \frac{1}{2}x^2\right) \ln 2 \Big|_{x=-1}^{x=2} \\ &= (4 \ln 2) - \left(-\frac{1}{2} \ln 2\right) \\ &= \frac{9}{2} \ln 2 \approx 3.12. \end{aligned}$$

We could have computed $\int_0^1 \int_{-1}^2 \frac{1+x}{1+y} dx dy$ instead and gotten the same final result. Try it!

MATLAB Examples

s804x05 [804/5 revisited]

Calculate the iterated integral $\int_0^4 \int_0^2 x\sqrt{y} dx dy$.

Solution

Here we replicate the work we did with our TI-89. Nested MATLAB **int** commands produce the exact answer directly, whereas a MATLAB version of the **smi** command I wrote shows you the steps. Here is a diary file.

```

%
%
% Stewart 804/5
%
%
syms x y
f = x*sqrt(y);
%
exact_answer = int(int(f, x,0,2), y,0,4)

exact_answer =

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floated = eval(exact_answer)
floated =
    10.6667
%
% Step-by-step!
exact_answer = smi(f, [x 0 2; y 0 4]);
STEPWISE (MULTIPLE) INTEGRATION!

Antiderivative w.r.t. x:

                2 1/2
            1/2 x y

When x = 2:

                1/2
            2 y

When x = 0:

                0

Difference:

                1/2
            2 y

Antiderivative w.r.t. y:

                3/2
            4/3 y

When y = 4:

                32/3

When y = 0:

                0

Difference:

                32/3
Answer (above) and approximaton (below)
    10.6667

```

s804x10

Calculate the iterated integral $\int_0^{\pi/2} \int_0^{\pi/2} \sin(x+y) dy dx$.

Solution

Let's rapidly dispatch this one with nested **int** commands. On your TI-89, just select the appropriate template off the **Muint** menu and fill in the blanks!

```

%
% Stewart 804/10
%
syms x y
exact_answer = int(int(sin(x+y), y,0,pi/2), x,0,pi/2)

exact_answer =

2

%
echo off; diary off

```

s804x26

Find the volume of the solid lying under the circular paraboloid $z = f(x, y) = x^2 + y^2$ and above the rectangular region $R = [-2, 2] \times [-3, 3]$.

Solution

Let's say the unit of length is the meter. Then volume is

$$\iint_R f(x, y) dA = \int_{-3}^3 \int_{-2}^2 x^2 + y^2 dx dy = 104 \text{ m}^3.$$

```

%
% Stewart 804/26
%
syms x y
exact_answer = int(int(x^2+y^2, x,-2,2), y,-3,3)

exact_answer =

104

%
echo off; diary off
Stewart 804/26

```

