Spring 2004 Math 253/501–503
13 Multiple Integrals
13.3 Double Integrals over General Regions
Tue, 24/Feb ©2004, Art Belmonte

Summary

Nonrectangular regions

- A Type I region has the form

  $$D = \{(x, y) : a \le x \le b, g_1(x) \le y \le g_2(x)\}.$$

  It is a region in the xy-plane bounded on the left by the vertical line $x = a$, on the right by the vertical line $x = b$, below by the curve $y = g_1(x)$, and above by the curve $y = g_2(x)$. The double integral of the function $f(x, y)$ over D may be computed as $\int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx$.

- A Type II region has the form

  $$D = \{(x, y) : c \le y \le d, h_1(y) \le x \le h_2(y)\}.$$

  It is a region in the xy-plane bounded on the below by the horizontal line $y = c$, above by the horizontal line $y = d$, on the left by the curve $x = h_1(y)$, and on the right by the curve $x = h_2(y)$. The double integral of the function $f(x, y)$ over D may be computed as $\int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy$.

- NOTES: The aforementioned "curves" may be lines. Also, vertical or horizontal boundaries may actually collapse to a single point. Finally, some regions are of both types.

- Let $f(x, y) \ge 0$ be continuous on D, a Type I or Type II region. Then the volume below the surface $z = f(x, y)$ and above the region D in the xy-plane is $\iint_D f(x, y)\, dA$.

Properties of double integrals

The familiar properties of single integrals carry over to double integrals. Let c be a real constant and let f and g be defined on $D = D_1 \cup D_2$, the union of two regions of Type I and/or Type II.

- $\iint_D f + g\, dA = \iint_D f\, dA + \iint_D g\, dA$
- $\iint_D cf\, dA = c \iint_D f\, dA$
- If $f \ge g$ on D, then $\iint_D f\, dA \ge \iint_D g\, dA$.
- $\iint_D f\, dA = \iint_{D_1} f\, dA + \iint_{D_2} f\, dA$
- $\iint_D 1\, dA = A = A(D)$, the area of D
- If $m \le f \le M$ on D, then $mA \le \iint_D f\, dA \le MA$.

Hand Examples

Again, use the TAMUCALC **Muint** menu and the **smi** command to speed up your work. Try a few by hand to keep in practice.

812/7

Evaluate the double integral $\iint_D xy\, dA$ over the region $D = \left\{(x, y) : 0 \le x \le 1, x^2 \le y \le \sqrt{x}\right\}$.

Solution

Here are the steps involved in the multiple integration. The fact that some limits of integration are symbols has no bearing on the mechanistic dispatch of the integral. We (or MATLAB or your TI-89) go through the same procedure as for rectangles. Both an exact answer and an approximation are provided.

$$\begin{aligned} \int_0^1 \int_{x^2}^{\sqrt{x}} xy\, dy\, dx &= \int_0^1 \tfrac{1}{2} xy^2 \Big|_{y=x^2}^{y=\sqrt{x}} dx \\ &= \int_0^1 \tfrac{1}{2}^2 - \tfrac{1}{2} x^5\, dx \\ &= \tfrac{1}{6} x^3 - \tfrac{1}{12} x^6 \Big|_{x=0}^{x=1} \\ &= \left(\frac{1}{6} - \frac{1}{12}\right) - (0) = \frac{1}{12} \approx 0.083 \end{aligned}$$

812/10

Evaluate the double integral $\iint_D x \sin y\, dA$ over the region $D = \{(x, y) : 0 \le y \le \pi/2, 0 \le x \le \cos y\}$.

Solution

You know the drill, people.

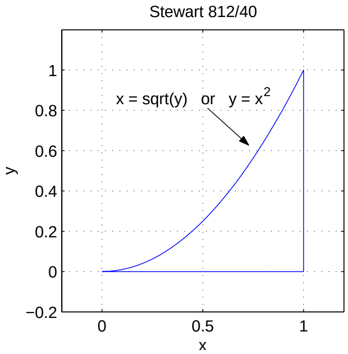
$$\begin{aligned}
\int_0^{\pi/2}\int_0^{\cos y} x\sin y\,dx\,dy &= \int_0^{\pi/2} \tfrac{1}{2}x^2\sin y\Big|_{x=0}^{x=\cos y} dy \\
&= \int_0^{\pi/2} \tfrac{1}{2}(\cos y)^2\sin y - 0\,dy \\
&= \left(-\tfrac{1}{6}(\cos y)^3\right)\Big|_{x=0}^{y=\pi/2} \\
&= (0)-\left(-\frac{1}{6}\right) \\
&= \frac{1}{6}\approx 0.17
\end{aligned}$$

812/40

Evaluate the integral $\int_0^1\int_{\sqrt{y}}^1 \sqrt{x^3+1}\,dx\,dy$ by reversing the order of integration.

Solution

1. First draw a picture of the region of integration!



2. Now set up and compute your new integral.

$$\begin{aligned}
\int_0^1\int_0^{x^2}\sqrt{x^3+1}\,dy\,dx &= \int_0^1 y\sqrt{x^3+1}\Big|_{y=0}^{y=x^2} dx \\
&= \int_0^1 \left(x^3+1\right)^{1/2} x^2\,dx \\
&= \tfrac{2}{9}\left(x^3+1\right)^{3/2}\Big|_{x=0}^{x=1} \\
&= \tfrac{2}{9}\left(2\sqrt{2}\right)-\tfrac{2}{9} = \tfrac{4}{9}\sqrt{2}-\tfrac{2}{9}\approx 0.41
\end{aligned}$$

MATLAB Examples

s812x24

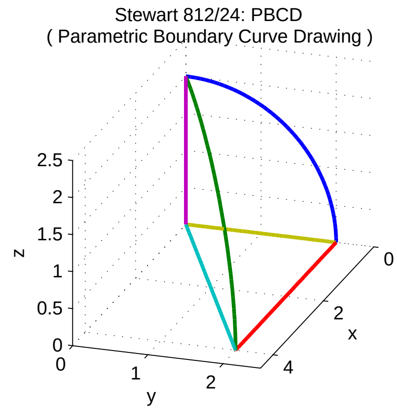
Find the volume of the solid in the first octant bounded by the circular cylinder $y^2+z^2=4$ and the planes $x=2y$, $x=0$, and, $z=0$. (Make sketches beforehand.)

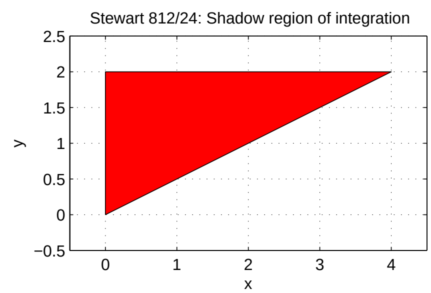
Solution

Here are two figures. The first shows the boundary curves of the solid, whereas the second shows the region of integration, which is the projection of the solid onto the xy-plane in this instance. The volume is

$$\iint_D f(x,y)\,dA = \int_0^2\int_0^{2y}\sqrt{4-y^2}\,dx\,dy = \frac{16}{3} = 5\tfrac{1}{3}\approx 5.33\text{ m}^3.$$

The picture of the shadow region helps us to set up our limits of integration.





It's easy to draw the 2-D plot of the shadow region. With practice you can sketch the 3-D plot of the solid. Here is a diary file showing the computation of the volume integral.

```
%
% Stewart 812/24s
%
syms x y
f = sqrt(4-y^2);
%
volume = int(int(f, x,0,2*y), y,0,2)

volume =

16/3

floated = eval(volume)
floated =
    5.3333
%

echo off; diary off
```

For those who are interested, here are the MATLAB script M-files that produced the graphics shown above.

```
%:::::::::::::
% Stewart 812/24a
%:::::::::::::
t = linspace(0, pi/2, 50);
x1 = 0*t;  y1 = 2*cos(t); z1 = 2*sin(t);
x2 = 2*y1; y2 = y1;       z2 = z1;
x3 = x2;   y3 = 0*y1 + 2; z3 = 0*z1;
x4 = x3;   y4 = y1;       z4 = z3;
x5 = 0*x1; y5 = 0*y1;     z5 = z1;
x6 = x1;   y6 = y1;       z6 = z3;
plot3(x1,y1,z1, x2,y2,z2, x3,y3,z3, ...
      x4,y4,z4, x5,y5,z5, x6,y6,z6)
grid on; axis equal
axis([0 4.5 0 2.5 0 2.5])
%:::::::::::::
% Stewart 812/24b
%:::::::::::::
M = [0 0; 4 2; 0 2]
x = M(:,1); y = M(:,2);
fill(x,y,'r')
axis equal
axis([-0.5 4.5 -0.5 2.5])
grid on
```