

Spring 2004 Math 253/501–503

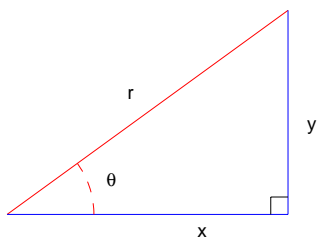
13 Multiple Integrals

13.4 Polar Coordinates

Thu, 26/Feb

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Summary



- Polar / rectangular relationships

$$x = r \cos \theta \quad y = r \sin \theta \quad r^2 = x^2 + y^2 \quad \tan \theta = \frac{y}{x}$$

- If $r > 0$, we march *forward* along the ray $\theta = \alpha$. If $r < 0$, we march *backward* along said ray.
- If $\theta > 0$, the angle is measured *counterclockwise* from the positive x -axis (the polar axis). If $\theta < 0$, the angle is measured *clockwise* from the positive x -axis. (This is the way the ancient Egyptians did it and this is the way we do it.)
- For a polar curve $r = f(\theta)$, form $x = r \cos \theta = f(\theta) \cos \theta$ and $y = r \sin \theta = f(\theta) \sin \theta$, then compute dy/dx parametrically as

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}.$$

- The arc length along a polar curve $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$ is

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} d\theta = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

Hand Examples

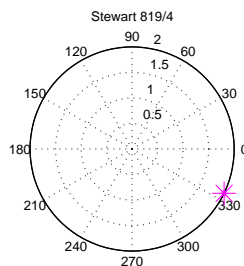
Coordinate/angle conversion commands are at the bottom of the TAMUCALC **Vect** menu. The **autopolar** command is near the top of the TAMUCALC **FVMD** menu.

819/4

Plot the point with polar coordinates $(r, \theta) = (2, -\pi/7)$. Then find two other pairs of polar coordinates of this point, one with $r > 0$ and one with $r < 0$.

Solution

Here's a plot of the point drawn with MATLAB's **polar** command.



Angles around the outermost circle are measured in degrees counterclockwise. The concentric circles depict varying radii r . Another pair of polar coordinates for this point with $r > 0$ is $(2, -\frac{\pi}{7} + 2\pi) = (2, \frac{13}{7}\pi)$. A different pair of polar coordinates for this point with $r < 0$ is $(-2, -\frac{\pi}{7} + \pi) = (-2, \frac{6}{7}\pi)$.

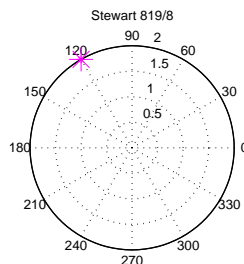
819/8

Plot the point with polar coordinates $(r, \theta) = (2, \frac{2}{3}\pi)$. Then find the Cartesian coordinates of the point.

Solution

The rectangular coordinates of the point depicted below are

$$(x, y) = (r \cos \theta, r \sin \theta) = \left(2 \cos \frac{2}{3}\pi, 2 \sin \frac{2}{3}\pi\right) = (-1, \sqrt{3}).$$



819/14

The Cartesian coordinates of a point are $(-1, -\sqrt{3})$. What are its polar coordinates (r, θ) , where $r > 0$ and $0 \leq \theta < 2\pi$?

Solution

Now $r = \sqrt{x^2 + y^2} = \sqrt{1 + 3} = 2$ and $\tan \theta = y/x = \sqrt{3}$. Since the point is in the third quadrant, we have $\theta = \pi + \frac{1}{3}\pi = \frac{4}{3}\pi$.

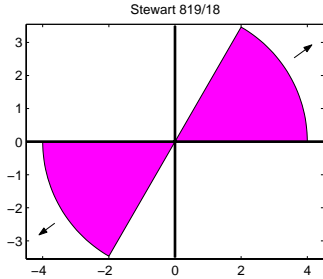
Hence $(r, \theta) = (2, \frac{4}{3}\pi)$.

819/18

Sketch the region in the plane consisting of the points whose polar coordinates satisfy $0 \leq \theta \leq \frac{\pi}{3}$.

Solution

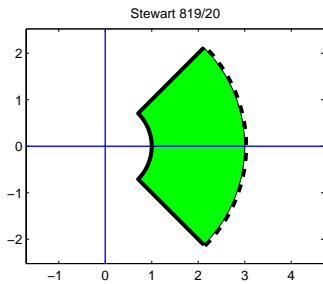
Recall that r can be negative as well as zero or positive. Therefore, the region is like two infinite slices of pie emanating from the pole (origin).

**819/20**

Sketch the region $1 \leq r < 3$, $-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$, in the plane.

Solution

This is a portion of a circular washer symmetric with the x -axis. Note that the $r = 3$ boundary is *not* included.

**819/30**

Find a Cartesian equation for the curve represented by the polar equation $r^2 = \theta$.

Solution

Take the tangent of each side of the polar equation, then substitute.

$$\begin{aligned} \tan(r^2) &= \tan \theta \\ \tan(x^2 + y^2) &= \frac{y}{x} \end{aligned}$$

819/36

Find a polar equation of the curve described by the Cartesian equation $x^2 - y^2 = 1$.

Solution

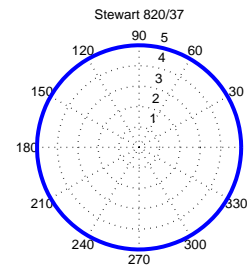
$$\text{We have } (r \cos \theta)^2 - (r \sin \theta)^2 = 1 \text{ or } r^2 = \frac{1}{\cos^2 \theta - \sin^2 \theta}.$$

820/37

Sketch the polar curve $r = 5$.

Solution

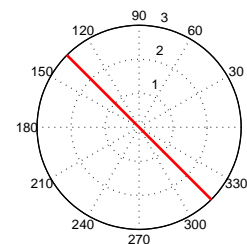
This is the circle of radius 5 centered at the pole (origin).

**820/38**

Sketch the polar curve $\theta = \frac{3}{4}\pi$.

Solution

This is a line through the pole (origin).

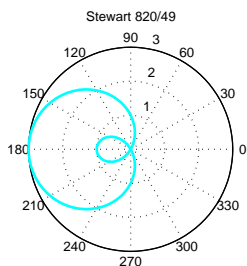
**MATLAB Examples****820/49**

Sketch the polar curve $r = 1 - 2 \cos \theta$.

Solution

Here is a polar plot. Use **autopolar** on your TI-89 to replicate it! (See the TAMUCALC Command Reference for details.)

```
%
% Stewart 820/49
t = linspace(0, 2*pi);
r = 1 - 2*cos(t);
polar(t, r, 'c');
%
echo off; diary off
```



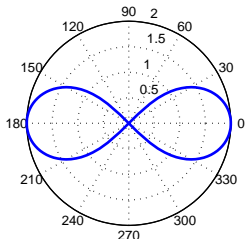
s820x55

Sketch the polar curve $r^2 = 4 \cos 2\theta$.

Solution

We'll plot $r = \pm 2\sqrt{\cos 2\theta}$ for $|\theta| \leq \frac{\pi}{4}$.

```
%
% Stewart 820/55
t = linspace(-pi/4, pi/4);
r1 = 2*sqrt(cos(2*t));
r2 = -r1;
polar(t,r1);
hold on
polar(t,r2)
%
echo off; diary off
Stewart 820/55: Look, Ma, it's the infinity symbol!
```



Example A

Find the slope of the tangent line to the cardioid $r = 1 + \cos \theta$ at the point corresponding to $\theta = \frac{\pi}{6}$.

Solution

First express x and y in terms of r and θ .

$$\begin{aligned} x &= r \cos \theta = (1 + \cos \theta) \cos \theta \\ y &= r \sin \theta = (1 + \cos \theta) \sin \theta \end{aligned}$$

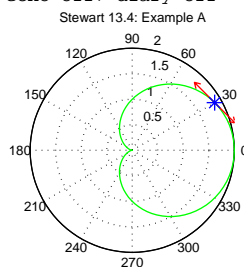
Now recall that $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$. We see that the slope at the desired point is $dy/dx = -1$. Here is a MATLAB diary file and a plot.

```
%
% Stewart 13.4/Example A
%
syms t
x = (1+cos(t)) * cos(t);
y = (1+cos(t)) * sin(t);
dy_dx = diff(y,t) / diff(x,t);
my_slope = simple(subs(dy_dx, t, sym(pi/6)))

my_slope =

-1

%
t = linspace(0, 2*pi);
r = 1 + cos(t);
polar(t,r,'g'); hold on
%
t0 = pi/6;
x0 = (1+cos(t0)) * cos(t0);
y0 = (1+cos(t0)) * sin(t0);
plot(x0,y0,'m*', 'MarkerSize', 12)
%
p = [x0; y0];
v = [-1; 1];
arrow(p,0.4*v,'r')
arrow(p,-0.4*v,'r')
%
echo off; diary off
Stewart 13.4: Example A
```



Example B

Find the arc length of the polar curve $r = \theta^2$, $0 \leq \theta \leq 2\pi$.

Solution

The length of this spiral is $\frac{8}{3} \left((\pi^2 + 1)^{3/2} - 1 \right) \approx 92.90$ cm.

```
%
% Stewart 13.4/Example B
%
syms t
r = t^2;
ig = simple(sqrt(r^2 + diff(r,t)^2))
```

```
ig =  
t*(t^2+4)^(1/2)  
L = int(ig, t, 0, 2*pi);  
pretty(L); eval(L)
```

$$\frac{8}{3} (\pi^2 + 1)^{3/2} - \frac{8}{3}$$

```
ans =  
92.8962  
%  
t = linspace(0, 2*pi);  
r = t.^2;  
polar(t,r)  
%
```

