Spring 2004 Math 253/501–503
13 Multiple Integrals
13.6 Applications of Double Integrals
Tue, 02/Mar ©2004, Art Belmonte

Summary

Let D be the region of integration, ρ the mass density, and σ the charge density. Note that $I_0 = I_x + I_y$. The center of mass is also called the centroid when the density is constant.

Application	2-D
differential	$dA = dx\,dy = r\,dr\,d\theta = J\,du\,dv$
measure	$A = \iint_D 1\,dA$ area
total mass	$m = \iint_D \rho\,dA$
electric charge	$Q = \iint_D \sigma\,dA$
moments	$M_x = \iint_D \rho y\,dA,\quad M_y = \iint_D \rho x\,dA$
center of mass	$[\bar{x}, \bar{y}] = \frac{1}{m} \iint_D \rho\,[x, y]\,dA$
moments of inertia	$I_x = \iint_D \rho y^2\,dA,\quad I_y = \iint_D \rho x^2\,dA$
radii of gyration	$\bar{\bar{x}} = \sqrt{I_y/m},\quad \bar{\bar{y}} = \sqrt{I_x/m}$

Hand / MATLAB Examples

832/2

Electric charge is distributed over the unit disk $x^2 + y^2 \le 1$ so that the charge density is $\sigma(x, y) = 1 + x^2 + y^2$ (measured in coulombs per square meter). Find the total charge on the disk.

Solution

The total charge is

$$\iint_D \sigma\,dA = \int_0^{2\pi} \int_0^1 \left(1 + r^2\right) \cdot r\,dr\,d\theta = \tfrac{3}{2}\pi \approx 4.71 \text{ coulombs.}$$

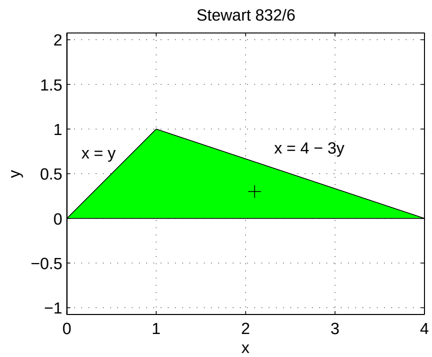
```
%
% Stewart 832/2
%
syms r t
exact_value = int(int((1+r^2) * r, r,0,1), t,0,2*pi);
pretty(exact_value)

                                    3/2 pi

floated = eval(exact_value)
floated =
    4.7124
%
echo off; diary off
```

832/6

Find the mass and center of mass of the lamina (flat plate) that occupies the triangular region D in the xy-plane with vertices $(0, 0)$, $(1, 1)$, and $(4, 0)$ and has variable density $\rho(x, y) = x$.

Solution

Here is a plot showing the region and its center of mass (+).



The mass is $m = \iint_D \rho\,dA = \int_0^1 \int_y^{4-3y} x\,dx\,dy = \frac{10}{3}$, whereas the center of mass is

$$\begin{aligned} CM = [\bar{x}, \bar{y}] &= \frac{1}{m} \iint_D \rho[x, y]\,dA \\ &= \frac{1}{10/3} \int_0^1 \int_y^{4-3y} x[x, y]\,dx\,dy \\ &= \left[\tfrac{21}{10}, \tfrac{3}{10}\right]. \end{aligned}$$

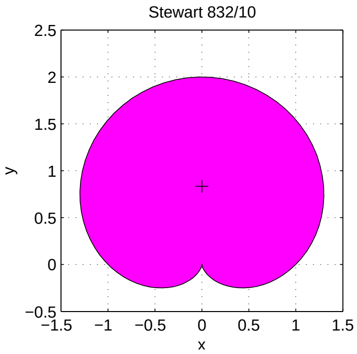
```
%
% Stewart 832/6
%
syms x y
m = int(int(x, x,y,4-3*y), y,0,1)

m =

10/3

CM = int(int([x,y]*x, x,y,4-3*y), y,0,1) / m

CM =

[ 21/10,  3/10]

%
echo off; diary off
```

832/10

Find the mass and center of mass of the lamina (flat plate) that is bounded by the cardioid $r = 1 + \sin\theta$ and has density $\rho = 2$.

Solution

Here is a plot showing the region and its center of mass (+).



The mass is $m = \iint_D \rho \, dA = \int_0^{2\pi} \int_0^{1+\sin\theta} 2 \cdot r \, dr \, d\theta = 3\pi$, whereas the center of mass is

$$\begin{aligned} CM = [\bar{x}, \bar{y}] &= \frac{1}{m} \iint_D \rho [x, y] \, dA \\ &= \frac{1}{3\pi} \int_0^{2\pi} \int_0^{1+\sin\theta} 2[r\cos\theta, r\sin\theta] \cdot r \, dr \, d\theta \\ &= \left[0, \tfrac{5}{6}\right]. \end{aligned}$$

```
%
% Stewart 832/10
%
syms r t
m = int(int(2 * r, r,0,1+sin(t)), t,0,2*pi)

m =

3*pi

CM = int(int([r*cos(t), r*sin(t)]*2 * r, ...
    r,0,1+sin(t)), t,0,2*pi) / m

CM =

[   0, 5/6]

%
echo off; diary off
```

832/21

A lamina (flat plate) with constant density $\rho = k$ occupies a square with vertices $(0, 0)$, $(a, 0)$, (a, a), and $(0, a)$. Find the moments of inertia I_x and I_y and the radii of gyration $\bar{\bar{x}}$ and $\bar{\bar{y}}$.

Solution

The mass is $m = \iint_D \rho \, dA = \int_0^a \int_0^a k \, dx \, dy = ka^2$.
The moments of inertia are

$$\begin{aligned} I_x = \iint_D \rho y^2 \, dA &= \int_0^a \int_0^a ky^2 \, dx \, dy = \tfrac{1}{3}ka^4, \\ I_y = \iint_D \rho x^2 \, dA &= \int_0^a \int_0^a kx^2 \, dx \, dy = \tfrac{1}{3}ka^4. \end{aligned}$$

The radii of gyration are

$$\begin{aligned} \bar{\bar{x}} &= \sqrt{I_y/m} = \tfrac{1}{3}\sqrt{3}a, \\ \bar{\bar{y}} &= \sqrt{I_x/m} = \tfrac{1}{3}\sqrt{3}a. \end{aligned}$$

```
%
% Stewart 832/21
%
syms a k x y positive
m = int(int(k, x,0,a), y,0,a); pretty(m)

                                       2
                                   k~ a~
Ix = int(int(y^2 * k, x,0,a), y,0,a); pretty(Ix)

                                         4
                                 1/3 k~ a~
Iy = int(int(x^2 * k, x,0,a), y,0,a); pretty(Iy)

                                         4
                                 1/3 k~ a~
x_bar_bar = sqrt(Iy/m); pretty(x_bar_bar)

                                      1/2
                                 1/3 3    a~
y_bar_bar = sqrt(Ix/m); pretty(y_bar_bar)

                                      1/2
                                 1/3 3    a~
%
echo off; diary off
```