

# Spring 2004 Math 253/501–503

## 13 Multiple Integrals

### 13.7 Surface Area

Tue, 02/Mar

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#### Summary

The **surface area** of an explicitly defined surface  $z = f(x, y)$ ,  $(x, y) \in D$ , whose first-order partial derivatives are continuous on  $D$  is given by  $S = \iint_D \sqrt{1 + f_x^2 + f_y^2} dA$ . If  $D$  is a polar region, switch to polar coordinates after setting up the integral.

#### Hand / MATLAB Examples

##### 834/2

Find the surface area of the part of the plane  $2x + 3y - z + 1 = 0$  that lies above the rectangle  $R = [1, 4] \times [2, 4]$ .

#### Solution

Let  $z = f(x, y) = 2x + 3y + 1$ . Then  $f_x = 2$ ,  $f_y = 3$ , and the surface area is

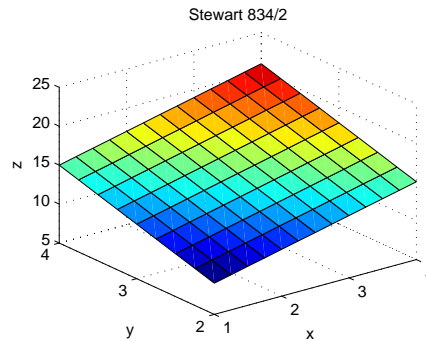
$$S = \int_2^4 \int_1^4 \sqrt{1 + 2^2 + 3^2} dx dy = 6\sqrt{14} \approx 22.45 \text{ cm}^2.$$

```
%
% Stewart 834/2
%
syms x y
f = 2*x + 3*y + 1;
grad_f = grad(f, [x y])

grad_f =
[ 2, 3]

S = int(int(sqrt(1 + len(grad_f)^2), x,1,4), y,2,4);
pretty(S)

eval(S)
ans =
    22.4499
%
echo off; diary off
```

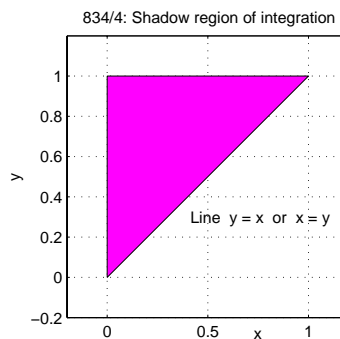


##### 834/4

Find the area of the part of the surface  $z = f(x, y) = x + y^2$  that lies above the triangle with vertices  $(0, 0)$ ,  $(1, 1)$ , and  $(0, 1)$ .

#### Solution

First,  $f_x = 1$  and  $f_y = 2y$ . Here is the region of integration.



Hence the surface area is

$$S = \int_0^1 \int_0^y \sqrt{1 + 1^2 + (2y)^2} dx dy = \frac{1}{2}\sqrt{6} - \frac{1}{6}\sqrt{2} \approx 0.99 \text{ cm}^2.$$

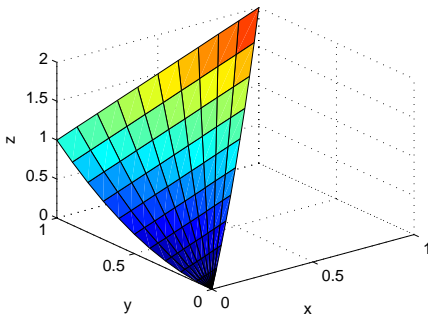
```
%
% Stewart 834/4
%
syms x y
f = x + y^2;
grad_f = grad(f, [x y])

grad_f =
[ 1, 2*y]

S = int(int(sqrt(1 + len(grad_f)^2), x,0,y), y,0,1);
pretty(S)

eval(S)
ans =
    0.9890
%
echo off; diary off
```

Stewart 834/4: Surface in space



**835/6**

Find the surface area of the part of the circular paraboloid  $z = f(x, y) = 4 - x^2 - y^2$  that lies above the  $xy$ -plane.

**Solution**

When the paraboloid intersects the  $xy$ -plane, we have

$$0 = z = 4 - x^2 - y^2.$$

Hence the curve of intersection is the circle  $x^2 + y^2 = 4, z = 0$ . This is the boundary of the region of integration, the circular disk  $D = \{(x, y) : x^2 + y^2 \leq 4\} = \{(r, \theta) : 0 \leq r \leq 2, 0 \leq \theta \leq 2\pi\}$ .

Now  $f_x = -2x$  and  $f_y = -2y$ . Thus the surface area is

$$S = \iint_D \sqrt{1 + 4x^2 + 4y^2} dA = \int_0^{2\pi} \int_0^2 \sqrt{1 + 4r^2} r dr d\theta = \frac{\pi}{6} (17\sqrt{17} - 1) \approx 36.18 \text{ cm}^2.$$

```
%
% Stewart 835/6
%
syms r t x y
f = 4 - x^2 - y^2;
grad_f = grad(f, [x y])

grad_f =
[ -2*x, -2*y]

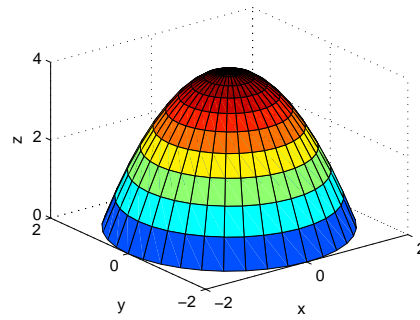
ig = sqrt(1 + len(grad_f)^2); pretty(ig)

          2      2 1/2
        (1 + 4 x  + 4 y )
S = int(int(sqrt(1 + 4*r^2) * r, r,0,2), t,0,2*pi);
pretty(factor(S))

          1/2
        1/6 pi (17 17  - 1)

eval(S)
ans =
    36.1769
%
echo off; diary off
```

Stewart 835/6



**835/14**

Find the area of the surface  $z = f(x, y) = 1 + x + y + x^2$ , above the rectangle  $-2 \leq x \leq 1, -1 \leq y \leq 1$ , in the  $xy$ -plane.

**Solution**

Now  $f_x = 1 + 2x$  and  $f_y = 1$ , whence

$$S = \int_{-2}^1 \int_{-1}^1 \sqrt{1 + (1 + 2x)^2 + 1^2} dy dx = 3\sqrt{11} + \ln\left(\frac{\sqrt{11} + 3}{\sqrt{11} - 3}\right) \approx 12.94 \text{ cm}^2.$$

```
%
% Stewart 835/14
%
syms x y
f = 1 + x + y + x^2;
grad_f = grad(f, [x y])

grad_f =
[ 1+2*x, 1]

S = int(int(sqrt(1 + len(grad_f)^2), y,-1,1), x,-2,1);
pretty(simple(S))

          1/2      1/2
        3 11  + log(-----)
                    1/2
                  -3 + 11

eval(S)
ans =
    12.9431
%
echo off; diary off
```

Stewart 835/14

