

# Spring 2004 Math 253/501–503

## 14 Vector Calculus

### 14.2 Line Integrals

Thu, 25/Mar ©2004, Art Belmonte

#### Summary

#### Framework

Let  $E \subset \mathbb{R}^n$  be a subset of  $n$ -dimensional space,  $f : E \rightarrow \mathbb{R}$  a scalar field and  $\mathbf{F} : E \rightarrow \mathbb{R}^n$  a vector field. Let  $C$  be a curve in  $n$ -D space, parameterized as  $\mathbf{g} : [a, b] \rightarrow \mathbb{R}^n$ , such that the range of  $\mathbf{g}$  is contained in  $E$ , so that the compositions  $f \circ \mathbf{g}$  and  $\mathbf{F} \circ \mathbf{g}$  are defined. (Typically  $n = 2$  or  $n = 3$  so that we're in the plane or 3-D space.) There are three types of line integrals we'll define.

#### Definitions

1. The line integral of  $f$  along  $C$  with respect to arc length is

$$\int_C f \, ds = \int_a^b f(\mathbf{g}(t)) \|\mathbf{g}'(t)\| \, dt.$$

2. The line integral of  $\mathbf{F}$  along  $C$  is

$$\int_C \mathbf{F} \cdot d\mathbf{g} = \int_a^b \mathbf{F}(\mathbf{g}(t)) \cdot \mathbf{g}'(t) \, dt.$$

If  $\mathbf{F}$  is a force field, this integral represents **work**.

3. The line integral of  $f$  with respect to the scalar variable  $v$  is

$$\int_C f \, dv = \int_a^b f(\mathbf{g}(t))v'(t) \, dt.$$

#### Applications of Line Integrals

Let  $\delta$  be the mass density and  $\sigma$  the charge density. The center of mass is also called the centroid when the density is constant.

Application	Line Integrals
scalar differential	$ds = \ \mathbf{g}'(t)\  \, dt$
measure	$L = \int_C 1 \, ds$ arc length
total mass	$m = \int_C \delta \, ds$
electric charge	$Q = \int_C \sigma \, ds$
moments <sup>1</sup>	$M_{yz} = \int_C x \delta \, ds$
center of mass	$[\bar{x}, \bar{y}, \bar{z}] = \frac{1}{m} \int_C [x, y, z] \delta \, ds$
moments of inertia <sup>2</sup>	$I_x = \int_C (y^2 + z^2) \delta \, ds$
radii of gyration	$\bar{x} = \sqrt{I_x/m}, \bar{y} = \sqrt{I_y/m}, \bar{z} = \sqrt{I_z/m}$

#### Table Notes

1. Other first-order moments are symmetrically defined.

$$M_{xz} = \int_C y \delta \, ds, \quad M_{xy} = \int_C z \delta \, ds$$

2. Other second-order moments are symmetrically defined.

$$I_y = \int_C (x^2 + z^2) \delta \, ds, \quad I_z = \int_C (x^2 + y^2) \delta \, ds$$

#### Hand Examples

On your TI-89, you'll use **lis** and **liv** to help you compute the first two types of line integrals given above, respectively.

#### 882/4 [directed line segment parameterization]

Evaluate the line integral  $\int_C xy \, ds$  where  $C$  is the directed line segment from  $A(-1, 1)$  to  $B(2, 3)$ .

#### Hand solution

1. From Chapter 11, the *directed* line segment from  $A$  to  $B$  is  $\mathbf{g}(t) = \mathbf{A} + t(\mathbf{B} - \mathbf{A}), 0 \leq t \leq 1$ .

$$\mathbf{g}(t) = [-1, 1] + t[3, 2] = [3t - 1, 2t + 1]$$

2. Form  $f(\mathbf{g}(t)) = f(3t - 1, 2t + 1) = (3t - 1)(2t + 1)$  via composition. Compute the vector derivative  $\mathbf{g}'(t) = [3, 2]$  and arc length differential  $ds = \|\mathbf{g}'(t)\| \, dt = \sqrt{13} \, dt$ .

3. Now put all the pieces together to compute the line integral.

$$\begin{aligned} \int_C f \, ds &= \int_a^b f(\mathbf{g}(t)) \|\mathbf{g}'(t)\| \, dt \\ &= \int_0^1 (3t - 1)(2t + 1) \cdot \sqrt{13} \, dt \\ &= \frac{3}{2} \sqrt{13} \approx 5.41 \end{aligned}$$

#### TI-89 solution

1. Define  $f$  and  $\mathbf{g}$  via **mf**. You may automatically parameterize the line segment via **pline2pt**, which gives the parametric line between two points. Test  $f$  and  $\mathbf{g}$  if desired:  $\mathbf{f}(\mathbf{x}, \mathbf{y})$  and  $\mathbf{g}(t)$ .

- **mf**([x,y], x\*y, "f")
- **mf**(t, pline2pt([-1,1], [2,3]), "g")

2. Compute the line integral via **lis** (Line Integral of a Scalar field w.r.t. arc length) on the **LSI** menu (which stands for Line and Surface Integrals).

- **lis**("f", "g", 0, 1) returns  $\frac{3}{2} \sqrt{13}$ , which floats to 5.41.

### 882/10 [helix parameterization]

Evaluate the line integral  $\int_C x^2 z ds$  where  $C$  is part of a helix:  $\mathbf{g}(t) = [\sin 2t, 3t, \cos 2t]$ ,  $0 \leq t \leq \frac{\pi}{4}$ .

#### Hand solution

- Form  $f(\mathbf{g}(t)) = f(\sin 2t, 3t, \cos 2t) = \sin^2 2t \cos 2t$  via composition. Compute  $\mathbf{g}'(t) = [2 \cos 2t, 3, -2 \sin 2t]$  and  $ds = \|\mathbf{g}'(t)\| dt = \sqrt{13} dt$ .
- Now put all the pieces together to compute the line integral.

$$\begin{aligned} \int_C f ds &= \int_a^b f(\mathbf{g}(t)) \|\mathbf{g}'(t)\| dt \\ &= \int_0^{\pi/4} \sin^2 2t \cos 2t \cdot \sqrt{13} dt \\ &= \frac{1}{6} \sqrt{13} \approx 0.601 \end{aligned}$$

**TI-89 solution** Define  $f$  and  $\mathbf{g}$  via **mf**, then invoke **lis**. [Note how  $f$  is a function of *three* variables so that the composition  $f(\mathbf{g}(t))$  is defined!]

- mf**([x,y,z], x^2\*z, "f")
- mf**(t, [sin(2\*t), 3\*t, cos(2\*t)], "g")
- lis**("f", "g", 0, pi/4) returns  $\frac{1}{6} \sqrt{13}$ , which floats to 0.601.

### 882/13

Evaluate the line integral  $\int_C x^3 y^2 z dz$  where  $C$  is the curve  $\mathbf{g}(t) = [2t, t^2, t^2]$ ,  $0 \leq t \leq 1$ .

#### “Hand” solution

- Form  $f(\mathbf{g}(t)) = f(2t, t^2, t^2) = (2t)^3 (t^2)^2 t^2 = 8t^9$  via composition. [Note: if you define  $f$  and  $\mathbf{g}$  via **mf**, then you may compute the composition via **compos**("f", "g", t).] Next, compute the scalar derivative  $dz = \frac{dz}{dt} dt = 2t dt$ .
- Now put all the pieces together to compute the line integral.

$$\begin{aligned} \int_C f dz &= \int_a^b f(\mathbf{g}(t)) z' dt \\ &= \int_0^{\pi/4} 8t^9 \cdot 2t dt \\ &= \frac{16}{11} \approx 1.45 \end{aligned}$$

### 882/37 [work]

Find the work done by the force field  $\mathbf{F}(x, y, z) = [xz, xy, yz]$  on a particle moving along the curve  $\mathbf{g}(t) = [t^2, -t^3, t^4]$ ,  $0 \leq t \leq 1$ .

#### Hand solution

- Form  $\mathbf{F}(\mathbf{g}(t)) = \mathbf{F}(t^2, -t^3, t^4) = [t^6, -t^5, -t^7]$  via composition (use **mf** and **compos**). Compute the vector derivative  $\mathbf{g}'(t) = [2t, -3t^2, 4t^3]$ , then form the dot product  $\mathbf{F}(\mathbf{g}(t)) \cdot \mathbf{g}'(t) = 5t^7 - 4t^{10}$  via **dot**.
- Now put all the pieces together to compute the line integral.

$$\begin{aligned} \text{work} &= \int_C \mathbf{F} \cdot d\mathbf{g} = \int_a^b \mathbf{F}(\mathbf{g}(t)) \cdot \mathbf{g}'(t) dt \\ &= \int_0^1 (5t^7 - 4t^{10}) dt \\ &= \frac{23}{88} \approx 0.26 \text{ J} \end{aligned}$$

**TI-89 solution** Define  $\mathbf{F}$  (or  $\mathbf{w}$ ) and  $\mathbf{g}$  via **mf**, then invoke **liv**. The calculator is not case sensitive, so we use  $\mathbf{w}$  (think of weight) to denote a vector field—as opposed to the  $f$  used to denote a scalar field. Here’s an alternative way to define functions using the **FCS** menu (Fields, Curves, and Surfaces). Warning: If you have an embedded TAMUCALC call (like **pline2pt**) in your functional expression, you *must* use **mf** instead (which always works).

- [x\*z, x\*y, y\*z]**→**w(x,y,z)**
- [t^2, -t^3, t^4]**→**g(t)**
- liv**("w", "g", 0, 1) returns  $\frac{23}{88}$ , which floats to 0.26 J.

### MATLAB Examples

#### 882/17

Evaluate the line integral  $\int_C \mathbf{w} \cdot d\mathbf{g}$  where  $\mathbf{w} = [x^2 y, -x y]$  and  $\mathbf{g}(t) = [t^3, t^4]$ ,  $0 \leq t \leq 1$ .

#### Solution

I’ve also implemented **liv** in MATLAB.

```
%
% Stewart 882/17
%
syms t x y
w = [x^2*y -x*y];
g = [t^3 t^4];
li = liv(w,g,0,1); pretty(li) % exact
```

```

-19
---
143

floated = double(li)
floated =
-0.1329
%
echo off; diary off

```

### 882/28 [circle parameterization]

Find the work done by the force field  $\mathbf{w}(x, y) = [x^2, xy]$  on a particle that moves once around the circle  $x^2 + y^2 = 4$ , oriented in a counterclockwise fashion. Explain the result.

#### Solution

Parameterize the circle as  $\mathbf{g}(t) = [2 \cos t, 2 \sin t]$ ,  $0 \leq t \leq 2\pi$ , then compute the line integral. The work is zero.

```

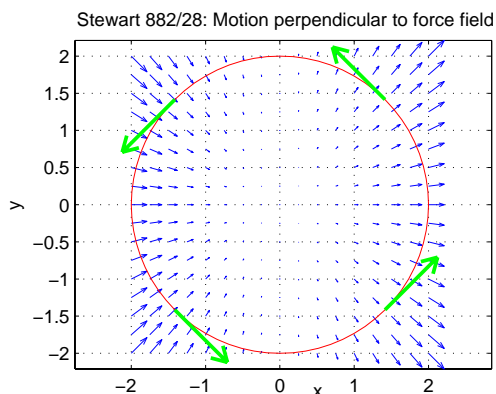
%
% Stewart 882/28
%
syms t x y
w = [x^2 x*y];
g = [2*cos(t) 2*sin(t)];
work = liv(w,g,0,2*pi); pretty(work)

0

%
echo off; diary off

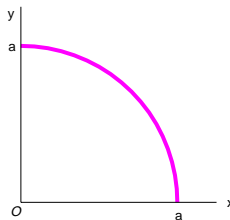
```

The direction of motion is perpendicular to the force field all along the circular path. So work  $= \int \mathbf{w} \cdot \mathbf{g}' dt = \int 0 dt = 0$ . The plot below illustrates this nicely.



### 882/30 [mass and center of mass along a thin wire]

Find the mass and center of mass of a thin wire in the shape of a quarter-circle  $x^2 + y^2 = a^2$ ,  $a > 0$ , in the first quadrant. The linear density along the wire is  $\delta = x + y$ .



#### Solution

Parameterize the circle as  $\mathbf{g}(t) = [a \cos t, a \sin t]$ ,  $0 \leq t \leq \frac{\pi}{2}$ , then compute the needed.

```

%
% Stewart 882/30
%
syms a positive
syms t x y
d = x+y;
g = [a*cos(t) a*sin(t)]

g =

[ a*cos(t), a*sin(t)]

m = lis(d,g,0,pi/2); pretty(m) % mass

2 a^2

%
w = [x y]*d

w =

[ (x+y)*x, (x+y)*y]

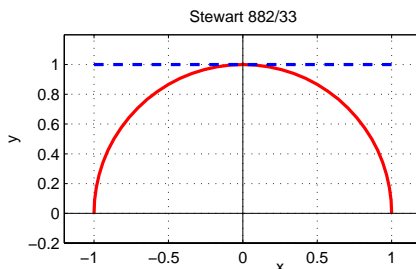
CM = simple( lis(w,g,0,pi/2)/m );
pretty(CM)

[1/8 a^2 (pi + 2) 1/8 a^2 (pi + 2)]
% You just computed two line integrals with
% respect to arc length in parallel, thereby
% pushing LIS beyond its design specifications!

```

### 882/33

Find the moments of inertia of a thin wire in the shape of the upper semicircle  $x^2 + y^2 = 1, y \geq 0$ , if the linear density at a point along the wire is proportional to its distance from the line  $y = 1$ ; that is,  $\delta = k(1 - y)$ .



## Solution

Parameterize the circle as  $\mathbf{g}(t) = [\cos t, \sin t]$ ,  $0 \leq t \leq \pi$ , then compute the needful. Once again, note the “parallel processing.”

```
%
% Stewart 882/33
%
syms k t x y
d = k*(1-y);
g = [cos(t) sin(t)]

g =

[ cos(t), sin(t)]

w = [x^2 y^2]*d

w =

[ k*(1-y)*x^2, k*(1-y)*y^2]

% Push LIS beyond its design specs!
moi = lis(w,g,0,pi);
Iy = moi(1); Ix = moi(2); I0 = sum(moi);
Ix_Iy_I0 = [Ix Iy I0]; pretty(Ix_Iy_I0)

[- 4/3 k + 1/2 pi k    - 2/3 k + 1/2 pi k    -2 k + pi k]
%
echo off; diary off
```

## Example A

Compute the line integral  $\int_C f ds$ , where  $f = \sqrt{x^2 + y^2 + z^4}$  and  $\mathbf{g}(t) = [e^t \cos t, e^t \sin t, t]$ ,  $-1 \leq t \leq 1$ .

## Solution

Look at the fur ball the cat coughed up! In other words, **lis** is unable to analytically compute the needed antiderivative and therefore returns an unevaluated integral. Only floating point will save you now. You have two choices (see below). [NOTE: When the TI-89 in AUTO mode fails to compute an integral exactly, it automatically starts to compute it numerically. This is precisely what we want it to do!]

1. Float the unevaluated integral with **double**.
2. Use **lisn**, which computes a line integral numerically using Simpson’s rule. Type “**help lisn**” at the MATLAB command prompt for details.

```
% Stewart 14.2/Example A
syms t x y z
f = sqrt(x^2 + y^2 + z^4);
g = [exp(t)*cos(t) exp(t)*sin(t) t];
li = lis(f,g,-1,1); pretty(li) % Exact FAILS,
 1
 /
 |
 |      4      1/2
 | ((exp(2 t~) + t~ ) (2 exp(2 t~) + 1)) dt~
 |
 /
-1
floated = double(li) % so float it with DOUBLE.
```

```
floated =
 6.0696
%
li1 = lisn(f,g,-1,1,40) % approximations
li1 =
 6.0696
li2 = lisn(f,g,-1,1,80)
li2 =
 6.0696
err = abs(li1-li2)
err =
 3.0376e-06
%
echo off; diary off
```

## Example B [another example of work]

This is Cooper’s Example 11.4 on pages 225–226 (*q.v.*) done more cleanly.

## Solution

This time the path is given as a table of  $(x, y)$ -coordinates. We use **pline2pt** to construct the directed line segments from point-to-point, compute the work on each portion with **livn** (as long as we’re floating things), then add up the results. Again, **livn** is the numerical version of **liv**.

```
%
% Stewart 14.2/Example B
% [a.k.a., Cooper’s Example 11.4
% via pline2pt and livn]
%
syms t x y
w = [x*cos(y) x+y];
X = [0 0.1 0.25 0.4 0.54 0.76 0.82 0.93 1];
Y = [0 0.005 0.0312 0.0800 0.1458 ...
      0.2888 0.3362 0.4352 0.5];
n = length(X); L = 0;
for k = 1:n-1
  g = pline2pt([X(k) Y(k)], [X(k+1) Y(k+1)]);
  L = L + livn(w,g,0,1,50);
end
Approximate value of line integral is:
 0.9353
```