Spring 2004 Math 253/501–503
14 Vector Calculus
14.5 Curl and Divergence
Thu, 08/Apr ©2004, Art Belmonte

Summary

Let $\mathbf{F}(x, y, z) = [P, Q, R]$ be a vector field and $f(x, y, z)$ be a scalar field.

Definitions

- The vector differential operator **del** is $\vec{\nabla} = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right]$.
- The **gradient** of f is $\vec{\nabla} f = [f_x, f_y, f_z]$.
- The **curl** of $\mathbf{F}$ is curl $\mathbf{F} = \vec{\nabla} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$ $= [R_y - Q_z, P_z - R_x, Q_x - P_y]$.
- The **divergence** of $\mathbf{F}$ is div $\mathbf{F} = \vec{\nabla} \cdot \mathbf{F} = P_x + Q_y + R_z$.
- The **laplacian** of f is $\vec{\nabla}^2 f = \vec{\nabla} \cdot \vec{\nabla} f = f_{xx} + f_{yy} + f_{zz}$.
- The **laplacian** of $\mathbf{F}$ is $\left[\vec{\nabla}^2 P, \vec{\nabla}^2 Q, \vec{\nabla}^2 R\right]$.

Theorems

- The curl of a gradient vector field is the zero vector. That is, curl $\left(\vec{\nabla} f\right) = \mathbf{0} = [0, 0, 0]$.
- If $\mathbf{F}$ is a vector field defined on $\mathbb{R}^3$ such that curl $\mathbf{F} = \mathbf{0}$ (the zero vector $[0, 0, 0]$), then $\mathbf{F}$ is conservative; i.e., $\mathbf{F} = \vec{\nabla} f$ for some scalar potential function f.
- If $\mathbf{F}$ is a vector field on $\mathbb{R}^3$ whose components have continuous second-order partial derivatives, then div curl $\mathbf{F} = 0$, the scalar zero.

Hand Examples

905/2

Find the curl & divergence of the vector field $\mathbf{F} = \left[x^2 y, yz^2, x^2 z\right]$.

Solution

- The curl of $\mathbf{F}$ is curl $\mathbf{F} = \vec{\nabla} \times \mathbf{F}$ or

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 y & yz^2 & x^2 z \end{vmatrix} = \left[-2yz, -2xz, -x^2\right].$$

- The divergence of $\mathbf{F}$ is div $\mathbf{F} = \vec{\nabla} \cdot \mathbf{F}$ or

$$\left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right] \cdot \left[x^2 y, yz^2, x^2 z\right] = 2xy + z^2 + x^2.$$

905/18

Is the vector field $\mathbf{F} = [xz, xy, yz]$ conservative? If so, find a scalar potential function f for $\mathbf{F}$.

Solution

Assume that $\mathbf{F}$, which is defined on $\mathbb{R}^3$, is conservative. Then it is the gradient of some scalar field f. By the first theorem, we have $\mathbf{0} =$ curl $\vec{\nabla} f =$ curl $\mathbf{F}$. But the curl of $\mathbf{F}$ is curl $\mathbf{F} = \vec{\nabla} \times \mathbf{F}$ or

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & xy & yz \end{vmatrix} = [z, x, y] \neq [0, 0, 0].$$

We thus have a contradiction. Accordingly, the vector field $\mathbf{F}$ is *not* conservative.

MATLAB Examples

905/6

Find the curl and divergence of $\mathbf{F} = \left[\sin x, \cos x, z^2\right]$.

Solution

In MATLAB you may use the **Curl** and **div** commands I wrote. (Cooper already had a different routine called **curl**.) There are also **curl** and **div** commands available on your TI-89 on the **FVMD** menu (Functions and Matrix & Vector Derivatives).

```
%
% Stewart 905/6
%
syms x y z
F = [sin(x)  cos(x)  z^2];
curl_F = Curl(F); pretty(curl_F)

                                  [0    0    -sin(x)]
div_F = div(F); pretty(div_F)

                                     cos(x) + 2 z
%
echo off; diary off
```

905/16

Is the vector field $\mathbf{F} = \left[x, e^y \sin z, e^y \cos z\right]$ conservative? If so, find a scalar potential function f for $\mathbf{F}$.

Solution

We see that curl $\mathbf{F} = \mathbf{0}$ on $\mathbb{R}^3$. So $\mathbf{F}$ is conservative by the second theorem in the Summary. We then use the **pot** command to yield the desired potential function f.

```
%
% Stewart 905/16
%
syms x y z
v = [x y z];
F = [x  exp(y)*sin(z)  exp(y)*cos(z)];
curl_F = Curl(F); pretty(curl_F)

                                 [0    0    0]
f = pot(F,v); pretty(f)

                                 2
                            1/2 x  + exp(y) sin(z)
%
echo off; diary off
```

906/38

Given the position vector field $\mathbf{r} = [x, y, z]$ and spherical radius variable $\rho = \sqrt{x^2 + y^2 + z^2}$, verify the identity $\overrightarrow{\nabla}^2 \rho^3 = 12\rho$, where $\overrightarrow{\nabla}^2$ is the laplacian.

Solution

We simply show that $\overrightarrow{\nabla}^2 \rho^3 - 12\rho = 0$. Done!

```
%
% Stewart 906/38
%
syms x y z
v = [x y z];
r = len(v)

r =

(x^2+y^2+z^2)^(1/2)

f = r^3;
% Left - Right = 0
lmrez = simple(laplacian(f) - 12*r)

lmrez =

0

%
echo off; diary off
```