Fall 2003 Math 308/501–502
1 Introduction
1.D Autonomous Equations, Stability, and the Phase Line
Fri, 05/Sep ©2003, Art Belmonte

Summary

Autonomous Equations A first-order **autonomous** differential equation is one of the form $dy/dt = y' = f(y)$. Notice that the derivative expression does not depend on the independent variable t. Accordingly, along a given horizontal line in the ty-plane (say $y = b$), the slopes are the same. Moreover, tangent line segments in the direction field along the vertical line $t = a$ are replicated along any other vertical line $t = a + k$. Not only does this give the direction field a rather uniform appearance, it also means that the DE lends itself readily to *qualitative* analysis of solution curves. That is, without even analytically solving the DE (if this is even possible), we can still say a lot about the behavior of its solutions.

Equilibria To do this, first find the zeros of f; i.e., the values y for which $f(y) = 0$. If $f(c) = 0$, then the *constant* function $y \equiv c$ is a solution of the DE, since $y' \equiv 0 = f(c) = f(y(t))$ for all t. The value c is called an **equilibrium point** and the constant function $y \equiv c$ is called an **equilibrium solution**. Looking at a direction field of an autonomous DE, its equilibrium solutions jump right out at you.

Phase Line The equilibrium points partition the vertical y-axis or number line, the so-called **phase line**, into intervals. By determining the *sign* of $y' = f$ on an interval ($+$ or $-$), we ascertain whether y is respectively increasing or decreasing on said interval. We signify this by respectively drawing upward ($\uparrow$) or downward ($\downarrow$) arrows along the phase line in each interval. Then we can immediately draw qualitative sketches of solution curves in the ty-plane *by hand* without even drawing a single tangent line segment. Amazing, but true!

Stability We distinguish three types of equilibrium points according to how nearby solution curves are attracted or repelled to the corresponding equilibrium solutions.

- An equilibrium point/solution $y = c$ for which *all* nearby solutions approach the point in the long run (i.e., $y(t) \to c$ as $t \to \infty$) is called an **asymptotically stable** equilibrium point (ASEP) or a **sink**; it "attracts" all nearby solutions. You identify it on the phase line from the fact that both adjacent arrows next to the point are directed toward it. We distinguish an ASEP with a cross ($\times$).
- An equilibrium point/solution for which *all* nearby solutions move away from it is called **unstable** equilibrium point (UEP) or a **source**; it "repels" all nearby solutions. On the phase line, both of the adjacent arrows next to the point are directed away from it. We mark a UEP with a circle ($\circ$).
- An equilibrium point to which some nearby solutions are attracted and from which others are repelled is called a **semistable** equilibrium point (SSEP) or a **node**. On the phase line, one adjacent arrow is directed toward the point, whereas the other is directed away from it. We mark an SSEP with a box ($\square$).

Hand Examples

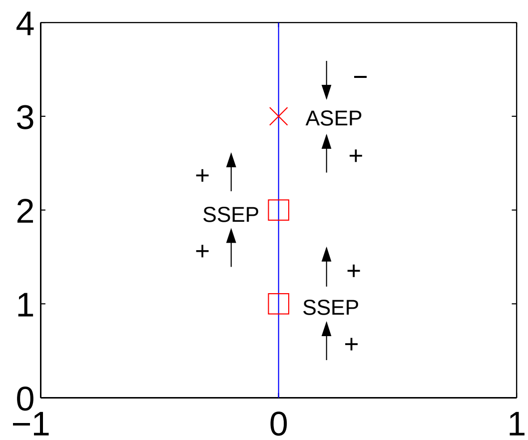
35/(e)

For the autonomous differential equation $y' = -(y-1)^{5/3}(y-2)^2(y-3)$, do the following by hand.

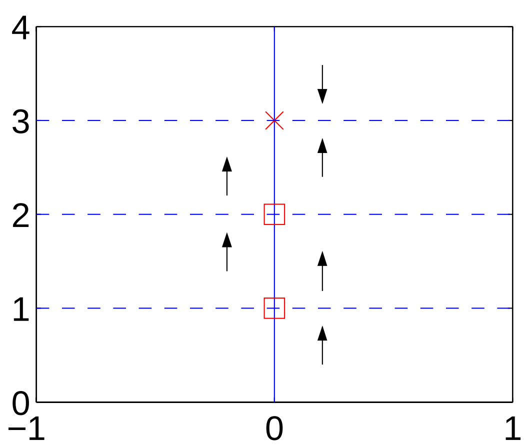
1. Determine the equilibrium points.
2. Develop the vertical y-axis phase line, appropriately distinguishing the equilibrium points.
3. Sketch the equilibrium solutions.
4. Sketch a few non-equilibrium solutions in each region of the ty-plane partitioned by the equilibrium solutions, including the particular solution that satisfies the initial condition $y(0) = 2.1$.

Solution

1. We solve $f(y) = -(y-1)^{5/3}(y-2)^2(y-3) = 0$ to obtain the equilibrium points: $y = 1$, $y = 2$, $y = 3$.
2. On the whiteboard in lecture, we developed this phase line.



3. Next, we sketched the equilibrium solutions, which are horizontal lines since these solutions are constant functions.



4. Finally, the phase line allows us to qualitatively draw non-equilibrium solutions in the various regions of the ty-plane partititioned by the equilibrium solutions. *I'll draw a rough sketch on the board for you so you get an idea of how this is done!*

MATLAB Examples

35/(e) [revisited]

Together we will corroborate the graph depicted in step #4 of the hand example above by using Polking's **dfield7** routine. I chose the rectangle $\{(t, y) : -1 \leq t \leq 10, -0 \leq y \leq 3.5\}$ so as to see the three equilibrium solutions. The choice of the t-range is arbitrary since the solution curves are horizontal translates of one another. I also chose to draw the curves only forward in time as opposed to both forward and backward (the default). On the **dfield7** Options menu, choose Solution Direction, then Forward.