Fall 2003 Math 308/501–502
2 First-Order Differential Equations
2.3 Linear Equations
Wed, 10/Sep ©2003, Art Belmonte

Summary

We saw in §1.1 that a first-order **linear** differential equation has the form $a_1(x)\dfrac{dy}{dx} + a_0(x)y = b(x)$. The **coefficients** a_0, a_1, b are constants or functions of the independent variable x alone; they do *not* depend on the dependent variable y. Moreover, note that both y and its derivative y' occur to the first power only. The equation is **homogeneous** if $b(x) = 0$ and **nonhomogeneous** if $b(x) \neq 0$.

Conventional procedure (CP)

1. You *MUST* put the DE in **standard linear form** (**SLF**):

$$y' + P(x)y = Q(x)$$

2. Construct an **integrating factor** $\mu(x) = \exp(\int P(x)\,dx)$. Note that *any* antiderivative $\int p(x)\,dx$ will do. So take the constant of integration to be zero.

3. Multiply the SLF by the integrating factor μ to obtain $\mu y' + \mu P y = \mu Q$. Notice *by design* that $(\mu y)' = \mu y' + \mu' y = \mu y' + \mu P y$ (via the Chain Rule). Putting these two together gives

$$(\mu(x)y(x))' = \mu(x)Q(x)$$

which is separable. That is, it's of the form $dw/dx = g(x)$ and we may use §2.2 techniques!

4. Antidifferentiate (i.e., indefinitely integrate) to obtain $\mu(x)y(x) = \int \mu(x)Q(x)\,dx + C$. We therefore have $y(x) = \dfrac{1}{\mu(x)}\displaystyle\int \mu(x)Q(x)\,dx + \dfrac{C}{\mu(x)}$, a **general solution** to the first-order linear DE.

Existence and Uniqueness of Solutions

Let y_0 be any real number. If $P(x)$ and $Q(x)$ are continuous on an interval (a, b) that contains x_0, then there is a *unique* solution to the IVP

$$\frac{dy}{dx} + P(x)y = Q(x), \quad y(x_0) = y_0$$

on the *entire* interval (a, b)—not just some smaller interval about x_0. (This is due to the nice structure of linear differential equations.)

Executive Summary of the CP!

1. SLF: $y' + Py = Q$.
2. IF: $\mu = \exp(\int P)$.
3. $\mu y' + \mu P y = \mu Q$, whence $(\mu y)' = \mu Q$.
4. Antidifferentiate; then isolate y.

This method works for both homogenous and nonhomogeneous first-order linear equtions. In the summary above, y is the dependent variable; the independent variable (unspecified) may be taken to be x. In other problems, t (time) is the independent variable and x or y (or some other letter) the dependent variable. Just be aware of the *structure!*

Hand Examples

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Find a general solution to the differential equation $\dfrac{dy}{dx} = \dfrac{y}{x} + 2x + 1$.

Solution

Let's go through the steps of the Conventional Procedure.

1. Put the DE into SLF: $y' - \dfrac{1}{x}y = 2x + 1$. Here $P(x) = -1/x$, the coefficient of y in the SLF.
2. Construct an integrating factor: $\mu(x) = \exp(\int P(x)\,dx)$ or $\exp(\int -1/x\,dx) = e^{-\ln x} = e^{\ln(x^{-1})} = x^{-1}$.
3. Multiply the SLF by μ: $x^{-1}y' - x^{-2}y = 2 + \dfrac{1}{x}$ or $(x^{-1}y)' = 2 + \dfrac{1}{x}$, as advertised!
4. Integrate to obtain $x^{-1}y = 2x + \ln|x| + C$. Thus $y = (2x + \ln|x| + C)x$ is a general solution.

(More examples appear on the next page.)

Example A

Find a general solution of $tx' = 4x + t^4$.

Solution

Let's go through the steps of the Conventional Procedure. (NOTE: In this problem, t is the independent variable and x is the dependent variable. Pay attention to the *structure*.)

1. Put the DE into SLF: $x' - \dfrac{4}{t}x = t^3$. Here $P(t) = -4/t$, the coefficient of x in the SLF.
2. Construct an integrating factor: $\mu(t) = \exp(\int P(t)\,dt)$ or $\exp(\int -4/t\,dt) = e^{-4\ln t} = e^{\ln(t^{-4})} = t^{-4}$.
3. Multiply the SLF by μ: $t^{-4}x' - 4t^{-5}x = 1/t$ or $(t^{-4}x)' = 1/t$, as advertised!
4. Integrate to obtain $t^{-4}x = \ln|t| + C$. Thus $x = t^4(\ln|t| + C)$ is a general solution.

Example B

Solve the IVP $y' + y = e^t$, $y(0) = 1$.

Solution

NOTE: In this problem, t is the independent variable and y is the dependent variable. Pay attention to the *structure*. Remember, with a 1st-order linear DE in SLF, we identify P as the coefficient of the dependent variable (not that of the derivative or the right-hand side of the DE).

1. The DE is already in SLF, with $P(t) = 1$.
2. Thus $\mu(t) = \exp(\int P(t)\,dt)$ or $\exp(\int 1\,dt) = e^t$.
3. Multiply the SLF by μ, yielding $e^t y' + e^t y = e^{2t}$ or $(e^t y)' = e^{2t}$.
4. Integrate to obtain $e^t y = \frac{1}{2}e^{2t} + C$. Thus $y = \frac{1}{2}e^t + Ce^{-t}$.
5. Now use the IC, $y(0) = 1$, to resolve the constant of integration C. We have $1 = y(0) = \frac{1}{2} + C$, whence $C = \frac{1}{2}$. Therefore, $y = \frac{1}{2}e^t + \frac{1}{2}e^{-t} = \cosh t$ is the unique solution of the IVP.

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Solve the IVP $\sin x \dfrac{dy}{dx} + y\cos x = x\sin x,\ \ y\left(\frac{\pi}{2}\right) = 2$.

Solution

What have we here? We're already at Step 3!

$$\begin{aligned}
(y\sin x)' &= x\sin x \\
y\sin x &= \sin x - x\cos x + C \\
y &= 1 - x\frac{\cos x}{\sin x} + \frac{C}{\sin x} \\
2 &= 1 + C \qquad \text{(using the IC)} \\
C &= 1
\end{aligned}$$

Therefore, $y = 1 - x\cot x + \csc x$.

MATLAB Examples

Example A [revisited]

Let's verify our hand solution via MATLAB's **dsolve** command. Note that the argument of the natural logarithm in the answer (written **log** in MATLAB) is t rather than $|t|$. This is because MATLAB actually regards the variables t and x as complex. If we regard them as real, then of course we must have $t > 0$, whence $|t| = t$ anyway!

```
sol = dsolve('t*Dx = 4*x + t^4', 't');
pretty(sol)
                 4
  (log(t) + C1) t
```

Example B [revisited]

Similary, we may use **dsolve** to verify the solution to the IVP.

```
sol = dsolve('Dy + y = exp(t)', ...
  'y(0)=1', 't');
pretty(sol)
  1/2 exp(t) + 1/2 exp(-t)
```