Fall 2003 Math 308/501–502
2 First-Order Differential Equations
2.4 Exact Equations
2.5 Special Integrating Factors
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Summary

A **differential form** in x and y is an expression of the type $w = P\,dx + Q\,dy$, where P and Q are functions of x and y. Simple forms dx and dy are called **differentials**. The differential form variant $P\,dx + Q\,dy = 0$ is simply another way of writing the differential equation $P + Q\dfrac{dy}{dx} = 0$. They have the same solutions, implicitly expressed as $F(x, y) = C$. The level curves (level sets in the xy-plane) so defined are termed **integral curves**.

Let $\mathbf{g} = [x, y]$. Recall from Calc 3 that the (total) differential of a continuously differentiable function f is the differential form

$$df = \vec{\nabla} f \cdot d\mathbf{g} = \left[f_x,\ f_y\right] \cdot [dx, dy] = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy$$

We say that a differential form $w = P\,dx + Q\,dy$ is **exact** if it is the differential of a continuously differentiable function f. (In this case the DE $w = P\,dx + Q\,dy = 0$ is referred to as an **exact differential equation**.) This is analogous (in the parlance of vector calculus) to the vector field $\mathbf{w} = [P, Q]$ being conservative if and only if it is the gradient of a scalar potential function f; i.e., $\mathbf{w} = \vec{\nabla} f$. Accordingly, we may utilize familiar hand and machine techniques from Calc 3 (in particular, Math 253) in this section.

Test for exactness

How can we test if a differential form $w = P\,dx + Q\,dy$ is exact? Phrased another way, how can we tell if the vector field $\mathbf{w} = [P, Q]$ is conservative? From Calc 3, we just test to see if $P_y = Q_x$ (under suitable hypotheses).

How to solve an exact differential equation

Find a potential function f for the vector field $\mathbf{w} = [P, Q]$ gleaned from the differential form $w = P\,dx + Q\,dy$. A general solution of the DE is then given by $f(x, y) = C$. We'll do this by hand and with MATLAB.

What if the differential form variant is NOT exact?

It may be possible to multiply $w = P\,dx + Q\,dy = 0$ by a suitable integrating factor (IF) μ such that the resulting equation $\mu P\,dx + \mu Q\,dy = 0$ *is* exact. We'll show you several scenarios in the hand examples.

Hand Examples

Example A

Determine if the DE $\dfrac{dy}{dx} = \dfrac{x}{x - y}$ is exact. If so, solve it.

Solution

We have $x\,dx + (y - x)\,dy = 0$. Hence $P = x$ and $Q = y - x$. Thus $P_y = 0$ and $Q_x = -1$. Hence $P_y \neq Q_x$. Accordingly, the DE is *not* exact.

Example B

Determine whether the DE $(1 - y\sin x)\,dx + \cos x\,dy = 0$ is exact. If so, solve it.

Solution

We have $P = 1 - y\sin x$ and $Q = \cos x$. Thus for any rectangular region in the xy-plane, $P_y = -\sin x = Q_x$. Accordingly, the DE is exact.

Construct a potential function f for the vector field $\mathbf{w} = [P, Q] = [1 - y\sin x, \cos x]$; i.e., find f such that $\mathbf{w} = \vec{\nabla} f$.

$\mathbf{w}$:	$1 - y\sin x$	$\cos x$
$\vec{\nabla} f$:	f_x	f_y
Antidiff	$x + y\cos x$	$y\cos x$
Harvest!	$f(x, y) =$	$x + y\cos x$

Therefore, $x + y\cos x = C$ or $y = \dfrac{C - x}{\cos x}$.

Example C

Show that the DE $(y^2 - xy)\,dx + x^2\,dy = 0$ is not exact, but that multiplying by the integrating factor (IF) $\mu = \dfrac{1}{xy^2}$ makes the resulting equation exact. Finally, solve the new equation.

Solution

We have $P_y = 2y - x \neq 2x = Q_x$. The original DE is not exact.

Multiplying the DE by $\mu = \dfrac{1}{xy^2}$ yields $(x^{-1} - y^{-1})\,dx + xy^{-2}\,dy = 0$. Now $P_y = y^{-2} = Q_x$. Hence the new equation is exact.

Construct a potential function f for the (new) vector field $\mathbf{w} = [P, Q] = \left[x^{-1} - y^{-1}, xy^{-2}\right]$.

w:	$x^{-1} - y^{-1}$	xy^{-2}
$\vec{\nabla} f$:	f_x	f_y
Antidiff	$\ln|x| - \frac{x}{y}$	$-\frac{x}{y}$
Harvest!	$f(x, y) =$	$\ln|x| - \frac{x}{y}$

A general solution is thus $\ln|x| - \dfrac{x}{y} = C$ or $y = \dfrac{x}{K + \ln|x|}$.

Example D

The DE $y\,dx + (x^2y - x)\,dy = 0$ has an IF $\mu = \mu(x)$ that is a function of x alone. Find it and solve the resulting DE.

Solution

Provided that $h(x) = \dfrac{P_y - Q_x}{Q}$ is *indeed* a function of x alone, an IF is given by $\mu = \exp(\int h(x)\,dx)$.

Now $h(x) = \dfrac{1 - (2xy - 1)}{x^2y - x} = \dfrac{2 - 2xy}{x^2y - x} = \dfrac{2(1 - xy)}{-x(1 - xy)} = -\dfrac{2}{x}$.
Thus $\mu = \exp(\int -2/x\,dx) = e^{-2\ln x} = e^{\ln(x^{-2})} = x^{-2}$.

Multiplying the DE by x^{-2} gives $x^{-2}y\,dx + (y - x^{-1})\,dy = 0$, which is exact since $P_y = x^{-2} = Q_x$.

Via the antidiff/harvest hand technique, we see that a potential function for $\mathbf{w} = \left[x^{-2}y,\, y - x^{-1}\right]$ is $f(x, y) = \dfrac{y^2}{2} - \dfrac{y}{x}$.
Therefore, a solution of the DE is $\dfrac{y^2}{2} - \dfrac{y}{x} = C$.

Example E

The DE $2y\,dx + (x + y)\,dy = 0$ has an IF $\mu = \mu(y)$ that is a function of y alone. Find it and solve the resulting DE.

Solution

Provided that $z(y) = \dfrac{Q_x - P_y}{P}$ is *indeed* a function of y alone, an IF is given by $\mu = \exp(\int z(y)\,dy)$.

Now $z(y) = \dfrac{1 - 2}{2y} = -\dfrac{1}{2y}$. Thus
$\mu = \exp(\int -\frac{1}{2y}\,dy) = e^{-\frac{1}{2}\ln y} = e^{\ln(y^{-1/2})} = y^{-1/2}$.

Multiplying the DE by $y^{-1/2}$ gives
$2y^{1/2}\,dx + (xy^{-1/2} + y^{1/2})\,dy = 0$, which is exact since
$P_y = y^{-1/2} = Q_x$.

Via the antidiff/harvest hand technique, a potential function for $\mathbf{w} = \left[2y^{1/2},\, xy^{-1/2} + y^{1/2}\right]$ is $f(x, y) = 2xy^{1/2} + \frac{2}{3}y^{3/2}$.
Therefore, a solution of the DE is $2x\sqrt{y} + \frac{2}{3}y^{3/2} = C$.

Example F

Find the family of curves that is orthogonal (perpendicular) to the family of parabolas defined by the equation $y^2 = cx$ and provide a sketch depicting the orthogonality of the two families.

Solution

Let $F(x, y) = y^2/x - c = 0$. Recall from Calc 3 that
$dy/dx = -F_x/F_y = -\dfrac{-y^2/x^2}{2y/x} = \dfrac{y}{2x}$. Thus the orthogonal family must satisfy $\dfrac{dy}{dx} = -\dfrac{2x}{y}$. (Think of negative reciprocal slopes from high school.) This is a separable equation. We have $\int y\,dy = \int -2x\,dx$, whence $\frac{1}{2}y^2 = -x^2 + A$ or $2x^2 + y^2 = K$, a family of ellipses. (For a graph depicting these orthogonal families, see the ***MATLAB Examples***.)

MATLAB Examples

Example A [revisited]

Will that dog hunt? In other words, does it have potential? Here we use the **pot** (potential) command I wrote to see if the vector field $\mathbf{w} = [x, y - x]$ is conservative; i.e, if it has a potential function f such that $\mathbf{w} = \vec{\nabla} f$. The command returns 0 (zero), signifying that the vector field is *not* conservative, that there is *no* potential function for **w**, and hence that the DE is *not* exact.

```
%
% NSS4-2.4/Example A: This dog won't hunt!
%
syms x y
w = [x, y-x]; v = [x, y];
no_potential_function = pot(w,v)
no_potential_function =
     0
```

Example B [revisited]

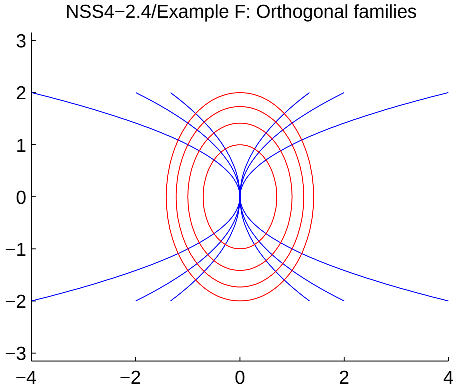
We first check our solution with MATLAB's **dsolve** command. It agrees with our explicit general solution. Then, mimicking our hand work, we see this time that the differential equation *is* exact since the **pot** command returns a potential function.

```
%
% NSS4-2.4/Example B: Potential function returned.
%
sol = dsolve('1 - y*sin(x) + cos(x)*Dy = 0', 'x');
pretty(sol)

                                    -x + C1
                                    -------
                                    cos(x)
%
```

```
syms x y
w = [1-y*sin(x), cos(x)]; v = [x, y];
our_potential_function = pot(w,v)

our_potential_function =

x+y*cos(x)

%
echo off; diary off
```

Example F [revisited]

Here is a graph depicting the family of parabolas together with the orthogonal family of ellipses.



This plot was produced with the following MATLAB M-file.

```
%
delete s24eF.txt; diary s24eF.txt
clear; clc; close all; echo on
%
% NSS4-2.4/Example F: Orthogonal families of curves
%
% First family
%
y = linspace(-2, 2, 100);
figure
hold on
for c=[-3,-2,-1]
    x = y.^2 / c;
    plot(x,y)
    echo off
end
echo on
for c=[1,2,3]
    x = y.^2 / c;
    plot(x,y)
    echo off
end
%
% Second family
%
G = inline('2*x.^2 + y.^2', 'x', 'y');
x = linspace(-4, 4, 100);
y = linspace(-2, 2, 100);
[X,Y] = meshgrid(x,y);
Z = G(X,Y);
contour(X,Y,Z, [1,2,3,4], 'r')
axis equal
%
echo off; diary off;
```