Fall 2003 Math 308/501–502
3 Mathematical Models
3.4 Newtonian Mechanics
Fri, 19/Sep ©2003, Art Belmonte

Summary

We'll use Newton's theory of motion from physics to model linear motion. Our starting point is Newton's second law,

$$F = ma$$

which says that force = mass × acceleration. We take into account gravitational force and possibly air resistance.

Hand Example

Example A

A mass of $0.2 = \frac{1}{5}$ kg is released from rest. [This means that its initial velocity is zero; i.e., $v(0) = 0$.] As the object falls, air provides a resistance proportional to the velocity. Specifically, the resistance force is $R(v) = -0.1v = -\frac{1}{10}v$, where velocity is measured in m/s. If the mass is dropped from a height of 50 m, what is its velocity when it hits the ground?

Solution

Velocity Let x be the distance of the object above the ground, so that the upward direction is positive. There are two forces acting on the object.

1. The gravitational force, $-mg$, is pulling the mass downward. Here $g = 9.8$ m/s^2. Below we'll use $9.8 = \frac{98}{10} = \frac{49}{5}$ (exact rationals).
2. Air resistance acts in the direction opposite to motion. Since the mass is falling downward, air resistance acts *upward*. Since a downward falling mass has negative velocity, the force of air resistance is $-\frac{1}{10}v$, which is *positive*, as required.

Apply Newton's second law to obtain $-mg - \dfrac{1}{10}v = F = ma = m\dfrac{dv}{dt}$, whence (recalling $m = \frac{1}{5}$)

$$\begin{aligned} \frac{dv}{dt} &= -g - \frac{1}{10m}v \\ \frac{dv}{dt} &= -\frac{49}{5} - \frac{1}{2}v \\ v' + \frac{1}{2}v &= -\frac{49}{5} \end{aligned}$$

This is both a separable and a linear equation, so we may employ any of the techniques from §2.2 or §2.4 that we prefer.

Let's go through the steps of the Conventional Procedure (CP) for linear equations for practice since we'll actually use the separable techique in the *second* part of this problem!

1. The DE in SLF is $v' + \frac{1}{2}v = -\frac{49}{5}$. Here $p(t) = \frac{1}{2}$, the coefficient of v in the SLF.
2. Construct an integrating factor: $\mu(t) = \exp(\int p(t)\,dt)$ or $\exp(\int \frac{1}{2}\,dt) = e^{t/2}$.
3. Multiply the SLF by μ: $e^{t/2}v' + \frac{1}{2}e^{t/2}v = -\frac{49}{5}e^{t/2}$ or $(e^{t/2}v)' = -\frac{49}{5}e^{t/2}$.
4. Integrate to obtain $e^{t/2}v = -\frac{98}{5}e^{t/2} + C$. Thus $v = -\frac{98}{5} + Ce^{-t/2}$.
5. Now use the IC, $v(0) = 0$, to resolve the constant of integration C. We have $0 = v(0) = -\frac{98}{5} + C$, whence $C = \frac{98}{5}$. Therefore, $v = \frac{98}{5}(e^{-t/2} - 1)$.

We now know the velocity at an arbitrary time t, but we don't know *when* the mass hits the ground. Let's solve for the position $x(t)$ at an arbitrary time then find the time of impact!

Position Since velocity is the derivative of position, we have $\dfrac{dx}{dt} = v = \dfrac{98}{5}(e^{-t/2} - 1)$, a separable equation. Let's find a general solution, then use the initial position to resolve the constant of integration.

1. Integrating, we have $x(t) = \frac{98}{5}(-2e^{-t/2} - t) + K$.
2. We have $50 = x(0) = -\frac{196}{5} + K$, whence $K = \frac{446}{5}$. Therefore, $x(t) = \frac{98}{5}(-2e^{-t/2} - t) + \frac{446}{5}$.

Time of Impact When the mass hits the ground, its height is zero; i.e., $x(t) = 0$. Solve the following equation for t numerically using MATLAB (see the ***MATLAB Example***) or the numerical solver on your calculator.

$$0 = x(t) = \frac{98}{5}(-2e^{-t/2} - t) + \frac{446}{5}$$

We obtain a positive time of $t \approx 4.32$ s.

Velocity of Impact Now compute the velocity at this time, $v(4.32) \approx -17.34$ m/s, *voila!* (Again, see the ***MATLAB Example*** or use your calculator.)

(Please see the next page for MATLAB computations.)

Example C

[NOTE: Example B is in the MATLAB examples!]

The brakes of a car are applied when it is moving at 100 km/h (i.e., $27\frac{7}{9}$ m/s). They provide a constant deceleration (that is, negative acceleration) of 10 m/s^2. How far does the car travel before coming to a stop?

Solution

- Velocity is the derivative of acceleration. Solve the IVP $dv/dt = -10,\ v(0) = 27\frac{7}{9}$, to obtain $v = \frac{250}{9} - 10t$.
- Position is the derivative of velocity. Solve the IVP $dx/dt = v = \frac{250}{9} - 10t\ x(0) = 0$, to obtain $x = \frac{250}{9}t - 5t^2$.
- When the car has stopped, its velocity is zero. Solving $0 = v = \frac{250}{9} - 10t$ shows that this occurs at $t = \frac{25}{9} \approx 2.78$ s.
- Finally, evaluate $x\left(\frac{25}{9}\right)$ to see that the car has traveled $\frac{3125}{81} \approx 38.58$ m.

MATLAB Examples

Example A [revisited]

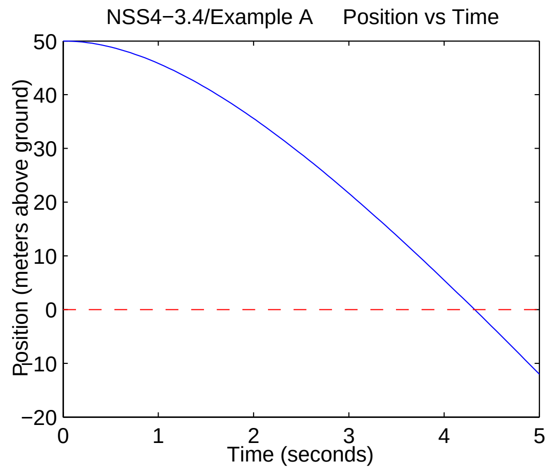
We solve for the velocity and position (distance fallen), then determine the time and velocity at impact. A graph of distance fallen vs time is provided for visual feedback and to choose a guess for MATLAB's **fzero** command.

```
%
% NSS4-3.4/Example A
%
% Solve for position; compute velocity.
%
syms t
x = dsolve('D2x + Dx/2 = -49/5', ...
  'x(0)=50', 'Dx(0)=0', 't');
pretty(x)

  - 196/5 exp(- 1/2 t) - 98/5 t + 446/5
v = diff(x,t); pretty(v)

  98/5 exp(- 1/2 t) - 98/5
%
% Solve for time of impact
% after obtaining guess from pic.
%
x = inline(vectorize(x), 't');
t = linspace(0, 5, 100);
x_vals = x(t);
plot(t,x_vals, t,zeros(size(x_vals)),'r--')
TOI = fzero(x, 4.3)
TOI =
    4.3204
%
% Velocity
%
v = inline(vectorize(v), 't');
velocity_at_impact = v(TOI)
velocity_at_impact =

  -17.3401
```



Example B

Lieutenant Jane of the Third Shock Army spearheads her unit's assault behind enemy lines. She bails out of the transport plane at an altitude of 10,000 ft, falls freely for 20 s, then opens her parachute. How long will it take her to reach the ground? Assume linear air resistance ρv ft/s^2, where $\rho = k/m$, taking $\rho = 0.15$ without the parachute and $\rho = 1.5$ with the parachute.

Solution

1. Let x be the distance the Lieutenant has fallen vertically. Start with Newton's second law.

$$\begin{aligned} F &= ma \\ mg - kv &= m\frac{dv}{dt} \\ \frac{dv}{dt} &= g - \frac{k}{m}v = g - \rho v \end{aligned}$$

2. Solve the IVP $v' = 32 - 0.15v,\ v(0) = 0$, to obtain her velocity at time t during free fall: $v = \frac{640}{3} - \frac{640}{3}e^{-3t/20}$.

3. Next, solve the IVP $x' = v = \frac{640}{3} - \frac{640}{3}e^{-3t/20},\ x(0) = 0$, to obtain the distance she has fallen by time t during free fall: $x = \frac{12800}{9}e^{-3t/20} + \frac{640}{3}t - \frac{12800}{9}$.

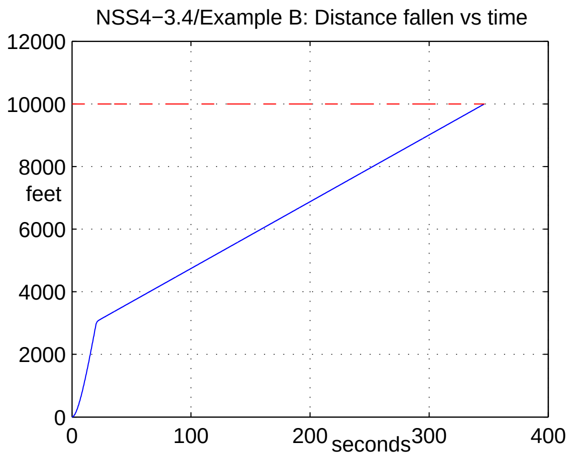
4. After 20 seconds, LT Jane has fallen $x(20) = 2915.3$ ft and is traveling at $v(20) = 202.71$ ft/s. This gives us initial conditions for a *second* pair of initial value problems. Solving these will give us her position and velocity while her parachute is open.

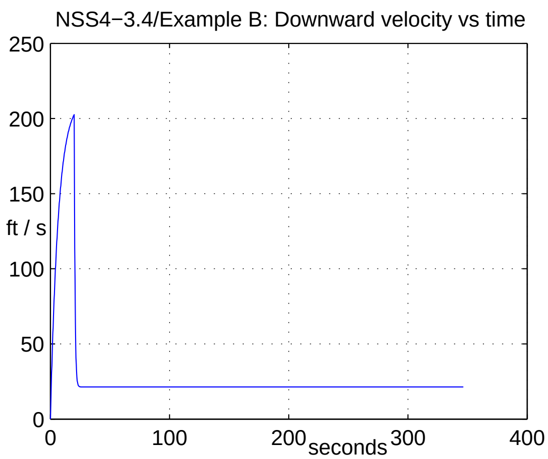
5. Reload, Ags! Solve the initial value problem $v' = 32 - 1.5v$, $v(20) = 202.71$, to obtain her velocity at time t (while her chute is open): $v = \frac{64}{3} + \frac{5441363}{30000} \times \frac{e^{-3t/2}}{e^{-30}}$.

6. Again, solve the IVP $x' = \frac{64}{3} + \frac{5441363}{30000} \times \frac{e^{-3t/2}}{e^{-30}}, \; x(20) = 2915.3$, to determine the distance she has fallen (while her chute is open): $x = -\frac{5441363}{45000} e^{30-\frac{3}{2}t} + \frac{64}{3}t + \frac{234855469}{90000}$. [Yes, troops, THIS is why we're using a computer!]

7. When the Lieutenant lands, she has fallen 10,000 ft vertically. Accordingly, solve $10000 = x(t) = -\frac{5441363}{45000} e^{30-\frac{3}{2}t} + \frac{64}{3}t + \frac{234855469}{90000}$ numerically to ascertain that it has taken her 346.4294 seconds, or 5 minutes 46.4 seconds to reach the ground.

As an added bonus, here are two graphs that visually illustrate Lieutenant Jane's descent.





Here is a diary file of the proceedings!

```
%
% NSS4-3.4/Example B
%
% FREE FALL
%
syms t
v = dsolve('Dv = 32 - 0.15*v', 'v(0)=0', 't');
pretty(v); v1 = v; % Velocity during free fall

                        640/3 - 640/3 exp(- 3/20 t)

%
x = dsolve(['Dx=' char(v1)], 'x(0)=0', 't');
pretty(x); x1 = x; % Distance fallen during free fall

                  12800/9 exp(- 3/20 t) + 640/3 t - 12800/9

%
v_rip = subs(v, t, 20) % Lieutenant Jane pulls
v_rip =
  202.7121
x_rip = subs(x, t, 20) % her chute's rip cord.
x_rip =
   2.9153e+03
%
% FALLING WITH PARACHUTE
%
v = dsolve('Dv = 32 - 1.5*v', ...
    ['v(20)=' num2str(v_rip)], 't');
pretty(v); v2 = v; % Velocity with chute deployed

                                5441363 exp(- 3/2 t)
                         64/3 + ------- ------------
                                 30000    exp(-30)

%
x = dsolve(['Dx=' char(v2)], ['x(20)=' num2str(x_rip)], 't');
pretty(x); x2 = x; % Distance fallen with chute deployed

                 5441363                                  234855469
               - ------- exp(- 3/2 t + 30) + 64/3 t + ---------
                  45000                                     90000
%
f = inline(vectorize(x2 - 10000), 't');
our_t = fzero(f, 350)
our_t =
  346.4294
minutes = floor(our_t / 60)
minutes =
     5
seconds = mod(our_t, 60)
seconds =
   46.4294
%
t = linspace(0, our_t, 2000);
v1 = eval(vectorize(v1));
x1 = eval(vectorize(x1));
v2 = eval(vectorize(v2));
x2 = eval(vectorize(x2));
v = (t<20) .* v1 + (t>=20) .* v2;
x = (t<20) .* x1 + (t>=20) .* x2;
x_10k = 10000 * ones(size(x));
plot(t,x, t,x_10k,'r--') % Plot of overall distance fallen
axis([0, 400, 0, 12000]); grid on
%
figure
plot(t,v); grid on % Plot of overall velocity
%
echo off; diary off;
```