Fall 2003 Math 308/501–502
4 Linear Second-Order Equations
4.3 Auxiliary Eqs with Complex Roots
Mon, 29/Sep ©2003, Art Belmonte

Summary

Terminology and Solutions

Consider the ODE $ay'' + by' + cy = 0$, where a, b, c are constants (with $a \neq 0$) and t is the independent variable (for specificity). The ODE has an associated **auxiliary** or **characteristic equation** $ar^2 + br + c = 0$. This quadratic equation has roots $r = r_1, r_2 = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, the discriminant of which is $b^2 - 4ac$. In the following , $c_1$and c_2 are arbitrary constants.

- When $b^2 - 4ac > 0$, the roots are real and distinct. In this case, $y_1(t) = e^{r_1 t}$ and $y_2(t) = e^{r_2 t}$ form a fundamental set of solutions to the ODE. A general solution is

  $$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}.$$

- When $b^2 - 4ac = 0$, there is a repeated or double root. In this case $y_1(t) = e^{rt}$ and $y_2(t) = te^{rt}$form a fundamental set of solutions to the ODE. A general solution is

  $$y(t) = c_1 e^{rt} + c_2 t e^{rt}.$$

- When $b^2 - 4ac < 0$, the roots are complex conjugate numbers, $r = \alpha \pm \beta i$, where $\alpha = -\frac{b}{2a}$, $\beta = \frac{\sqrt{b^2-4ac}}{2a}$, and $i = \sqrt{-1}$. Here $y_1(t) = e^{\alpha t}\cos\beta t$ and $y_2(t) = e^{\alpha t}\sin\beta t$ form a (real) fundamental set of solutions to the ODE.
  A general solution is

  $$y(t) = c_1 e^{\alpha t}\cos\beta t + c_2 e^{\alpha t}\sin\beta t.$$

  (NOTE: The real fundamental set of solutions are derived from the complex solution $e^{(\alpha+\beta i)t}$ via Euler's formula and then taking real and imaginary parts.)

Cauchy-Euler Equations

These are *variable*-coefficient differential equations of the form $ax^2\dfrac{d^2y}{dx^2} + bx\dfrac{dy}{dx} + cy = h(x),\ x > 0$, where a, b, c are constants. The substituion $x = e^t$ transforms this ODE into a *constant*-coefficient one, $a\dfrac{d^2y}{dt^2} + (b-a)\dfrac{dy}{dt} + cy = h\left(e^t\right)$. We solve this latter equation, then use the inverse substitution $t = \ln x$ to obtain a solution of the original ODE. (See 178/38 for a derivation and 179/40 for an example below.)

Hand Examples

Example A

Find a general solution of $y'' + 4y = 0$.

Solution

The characteristic equation is $0 = r^2 + 4 = 0$, whence $r = \pm 2i = 0 \pm 2i$. A general solution of the DE is $y(t) = e^{0t}(c_1\cos 2t + c_2\sin 2t) = c_1\cos 2t + c_2\sin 2t$.

Example B

Solve the IVP $\quad y'' - 2y' + 17y = 0, \quad y(0) = -2, \quad y'(0) = 3.$

Solution

The characteristic equation is $r^2 - 2r + 17 = 0$. The quadratic formula gives $r = \dfrac{2 \pm \sqrt{4-68}}{2} = 1 \pm 4i$. A general solution is $y(t) = e^t(c_1\cos 4t + c_2\sin 4t)$, the derivative of which is

$$y'(t) = e^t(4c_2\cos 4t - 4c_1\sin 4t) + e^t(c_1\cos 4t + c_2\sin 4t)$$

Employing the initial conditions, we have $-2 = y(0) = c_1$ and $3 = y'(0) = 4c_2 + c_1$. Thus $c_1 = -2$ and $c_2 = (3 - c_1)/4 = \frac{5}{4}$. The unique solution of the IVP is $y(t) = e^t\left(\frac{5}{4}\sin 4t - 2\cos 4t\right)$.

178/38

Show how the substitution $x = e^t$ changes the Cauchy-Euler equation $ax^2\dfrac{d^2y}{dx^2} + bx\dfrac{dy}{dx} + cy = h(x),\ x > 0$ into the ODE $a\dfrac{d^2y}{dt^2} + (b-a)\dfrac{dy}{dt} + cy = h\left(e^t\right)$.

Solution

(a) Using $x = e^t$ together with the Chain Rule, we have

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = \frac{dy}{dx}e^t = x\frac{dy}{dx}.$$

(b) Now repeat using the Chain and product rules!

$$\frac{d^2y}{dt^2} = x\frac{d^2y}{dx^2}\frac{dx}{dt} + \frac{dx}{dt}\frac{dy}{dx} = x^2\frac{d^2y}{dx^2} + \frac{dy}{dt}$$

whence $x^2\dfrac{d^2y}{dx^2} = \dfrac{d^2y}{dt^2} - \dfrac{dy}{dt}$.

(c) Using (a) and (b) with $x = e^t$, we have

$$\begin{aligned} ax^2\,\frac{d^2y}{dx^2} + bx\,\frac{dy}{dx} + cy &= h(x) \\ a\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) + b\,\frac{dy}{dt} + cy &= h\left(e^t\right) \\ a\,\frac{d^2y}{dt^2} + (b-a)\,\frac{dy}{dt} + cy &= h\left(e^t\right). \end{aligned}$$

179/40

Solve the Cauchy-Euler equation $x^2y''(x) + 7xy'(x) - 7y(x) = 0$.

Solution

According to 178/38 (*q.v.*), the substitution $x = e^t$ transforms the given equation into $y''(t) + 6y'(t) - 7y(t) = 0$. The characteristic equation of the latter is $0 = r^2 + 6r - 7 = (r-1)(r+7)$, whence $r = 1, -7$. Thus $y = c_1e^t + c_2e^{-7t} = c_1x + c_2x^{-7}$ (recalling that $t = \ln x$).

MATLAB Examples

Example A [revisited]

Find a general solution of $y'' + 4y = 0$.

Solution

You know the drill by now campers!

```
%
% NSS4-4.3/Example A
%
syms r
p = poly2sym([1 0 4], r); pretty(p)

                                                2
                                               r  + 4
r = solve(p, r)

r =

[  2*i]
[ -2*i]

%
y = dsolve('D2y + 4*y = 0', 't')

y =

C1*sin(2*t)+C2*cos(2*t)

%
echo off; diary off
```

Example B [revisited]

Solve the IVP $\quad y'' - 2y' + 17y = 0, \quad y(0) = -2, \quad y'(0) = 3.$

Solution

Matrix-vector techniques make our work easy!

```
%
% NSS4-4.3/Example B
%
syms r t
p = poly2sym([1 -2 17], r); pretty(p)

                                   2
                                  r  - 2 r + 17
r = solve(p, r)

r =

[ 1+4*i]
[ 1-4*i]

yf = [exp(t)*cos(4*t), exp(t)*sin(4*t)];
%
M = wron(yf, t); M = subs(M, t, 0)
M =
     1     0
     1     4
b = sym([-2; 3]);
c = M\b

c =

[  -2]
[ 5/4]

%
y = yf*c; pretty(y)

    -2 exp(t) cos(4 t) + 5/4 exp(t) sin(4 t)
%
sol = dsolve('D2y - 2*Dy + 17*y = 0', ...
    'y(0)=-2', 'Dy(0)=3', 't');
pretty(sol)

    -2 exp(t) cos(4 t) + 5/4 exp(t) sin(4 t)
%
echo off; diary off
```

179/40 [revisited]

Solve the Cauchy-Euler equation $x^2y''(x) + 7xy'(x) - 7y(x) = 0$.

Solution

We check our hand work with **dsolve**.

```
sol = dsolve('x^2*D2y + 7*x*Dy - 7*y = 0', 'x')
sol =
    C1*x  +  C2/x^7
```