

Spring 2003 Math 308/501–502
4 Linear Second-Order Equations
4.5 Method of Undetermined Coeffs
Wed, 01/Oct ©2003, Art Belmonte

Summary

The **method of undetermined coefficients** is dealt with in full generality in the Section 6.3 lecture, *q.v.* In this lecture handout, we'll restrict ourselves to examples of second-order equations.

By hand or with MATLAB, you're a winner!

The hand work involved in computing derivatives, substitution, collecting terms, and solving linear systems can be tedious and error-prone. It lends itself quite well, however, to machine power, as you'll observe in the *MATLAB Examples*.

You may have felt that the formulation for y_p given in Section 6.3 is overly complicated. Take solace in the fact that under that stated formulation there is only *one* case (the general case). What's more, there is *no* guesswork as to the correct *form* of y_p . With practice, you'll get it right every time!

Note that in all examples (hand or MATLAB), we carry the work through to a full general solution (or to the unique solution of an initial value problem). Please do this in your homework problems as well.

Hand Examples

Example A

Find a general solution of $y'' + 3y' - 18y = 18e^{2t}$.

Solution

1. The homogeneous equation $y'' + 3y' - 18y = 0$ has characteristic equation $0 = r^2 + 3r - 18 = (r - 3)(r + 6)$, with roots $r = 3, -6$. Hence $y_h = c_1e^{3t} + c_2e^{-6t}$. The forcing function is $f(t) = 18e^{2t}$, which is *not* a solution of the homogeneous equation. ("There is no interference.")
2. So let $y_p = ae^{2t}$. Then $y'_p = 2ae^{2t}$ and $y''_p = 4ae^{2t}$. Substituting into the nonhomogeneous equation gives

$$\begin{aligned} (4ae^{2t}) + 3(2ae^{2t}) - 18(ae^{2t}) &= 18e^{2t} \\ (-8a - 18)e^{2t} &= 0 = 0e^{2t} \end{aligned}$$

3. Equating coefficients of like entities, we have $-8a - 18 = 0$, whence $a = -\frac{9}{4}$. Thus $y_p = -\frac{9}{4}e^{2t}$ is our particular solution.

4. A general solution (verified by **dsolve**) is

$$y = y_p + y_h = -\frac{9}{4}e^{2t} + c_1e^{3t} + c_2e^{-6t}$$

Example B

Find a general solution of $y'' + 7y' + 10y = -4 \sin 3t$.

Solution

1. The homogeneous equation $y'' + 7y' + 10y = 0$ has characteristic equation $0 = r^2 + 7r + 10 = (r + 2)(r + 5)$, with roots $r = -2, -5$. Hence $y_h = c_1e^{-2t} + c_2e^{-5t}$. The forcing function is $f(t) = -4 \sin 3t$, which is *not* a solution of the homogeneous equation.
2. Let $y_p = a \cos 3t + b \sin 3t$. Then

$$\begin{aligned} y'_p &= -3a \sin 3t + 3b \cos 3t \\ y''_p &= -9a \cos 3t - 9b \sin 3t \end{aligned}$$

Thus $(-9a \cos 3t - 9b \sin 3t) + 7(-3a \sin 3t + 3b \cos 3t) + 10(a \cos 3t + b \sin 3t) = -4 \sin 3t$. Accordingly, we have

$$(a + 21b) \cos 3t + (4 + b - 21a) \sin 3t = 0 = 0 \cos 3t + 0 \sin 3t$$

3. Equating coefficients of like entities gives $a + 21b = 0$ and $4 - 21a + b = 0$, from which $a = \frac{42}{221}$, $b = -\frac{2}{221}$. For example, form the matrix system

$$\begin{bmatrix} 1 & 21 \\ -21 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ -4 \end{bmatrix}$$

or $\mathbf{M}\mathbf{c} = \mathbf{k}$. Then the MATLAB command $\mathbf{c} = \mathbf{M} \backslash \mathbf{k}$ solves the system. In the *MATLAB Examples*, I just used **solve**. Our particular solution is $y_p = \frac{42}{221} \cos 3t - \frac{2}{221} \sin 3t$.

4. A general solution (verified by **dsolve**) is

$$y = y_p + y_h = \frac{42}{221} \cos 3t - \frac{2}{221} \sin 3t + c_1e^{-2t} + c_2e^{-5t}$$

MATLAB Examples

First we revisit the two problems we did by hand. Then we'll examine a variety of problems that point out various subtleties.

I purposely left comments out of the code and solutions. Try to understand what is going on from the steps we did by hand in the first two examples. In time, you ought to be able to explain to yourself (or a classmate) what is going on in your own words.

If you have questions, however, please ask! (This is especially true when you're just starting out).

Example A, revisited [exponential forcing term]

Find a general solution of $y'' + 3y' - 18y = 18e^{2t}$.

Solution

```

%
% NSS4-4.5/Example A
%
syms a c1 c2 r t
y = sym('y(t)');
p = poly2sym([1 3 -18], r); pretty(p)

      2
      r  + 3 r - 18
r = solve(p)

r =

[ 3]
[-6]

yh = c1*exp(3*t) + c2*exp(-6*t);
%
L = diff(y,t,2) + 3*diff(y,t) - 18*y; pretty(L)

      / 2      \
      |d      |
      |--- y(t)| + 3 |-- y(t)| - 18 y(t)
      | 2      |
      \dt      /

yp = a*exp(2*t);
eq0 = subs(L - 18*exp(2*t), y, yp);
% Left - Right = 0
eq0 = simple(eq0 / exp(2*t))

eq0 =

-8*a-18

a = solve(eq0)

a =

-9/4

%
yp = subs(yp); pretty(yp)

- 9/4 exp(2 t)
check = subs(L, y, yp)

check =

18*exp(2*t)

%
y = yp + yh; pretty(y)

- 9/4 exp(2 t) + c1 exp(3 t) + c2 exp(-6 t)
sol = dsolve('D2y + 3*Dy - 18*y = 18*exp(2*t)', 't');
pretty(sol)

exp(-6 t) C2 + exp(3 t) C1 - 9/4 exp(2 t)
%
echo off; diary off

```

Example B, revisited [trigonometric forcing term]

Find a general solution of $y'' + 7y' + 10y = -4 \sin 3t$.

Solution

```

%
% NSS4-4.5/Example B
%
syms a b c1 c2 r t
y = sym('y(t)');
p = poly2sym([1 7 10], r); pretty(p)

      2
      r  + 7 r + 10
r = solve(p)

r =

[-2]
[-5]

yh = c1*exp(-2*t) + c2*exp(-5*t);
%
L = diff(y,t,2) + 7*diff(y,t) + 10*y; pretty(L)

      / 2      \
      |d      |
      |--- y(t)| + 7 |-- y(t)| + 10 y(t)
      | 2      |
      \dt      /

yp = a*cos(3*t) + b*sin(3*t);
eq0 = subs(L - (-4*sin(3*t)), y, yp);
eq0 = collect(eq0, cos(3*t));
eq0 = collect(eq0, sin(3*t))

eq0 =

(4+b-21*a)*sin(3*t)+(21*b+a)*cos(3*t)

[a b] = solve(4+b-21*a, 21*b+a) % Cut & paste equations!

a =

42/221

b =

-2/221

%
yp = subs(yp); pretty(yp)

      42
      --- cos(3 t) - 2/221 sin(3 t)
      221
check = subs(L, y, yp)

check =

-4*sin(3*t)

%
y = yp + yh; pretty(y)

      42
      --- cos(3 t) - 2/221 sin(3 t) + c1 exp(-2 t) + c2 exp(-5 t)
      221
sol = dsolve('D2y + 7*Dy + 10*y = -4*sin(3*t)', 't');
pretty(sol)

exp(-5 t) C2 + exp(-2 t) C1 + --- cos(3 t) - 2/221 sin(3 t)
      221
%
echo off; diary off

```

Example C [polynomial forcing term]

Find a general solution of $y'' + 5y' + 6y = 4 - t^2$.

Solution

```

%
% NSS4-4.5/Example C
%
syms a b c c1 c2 r t
y = sym('y(t)');
p = poly2sym([1 5 6], r); pretty(p)

      2
      r  + 5 r + 6
r = solve(p)

r =

[ -2]
[ -3]

yh = c1*exp(-2*t) + c2*exp(-3*t);
%
L = diff(y,t,2) + 5*diff(y,t) + 6*y; pretty(L)

      / 2      \
      |d      |
      |--- y(t)| + 5 |d      |
      | 2      |    \dt      /
      \dt      /

yp = a*t^2 + b*t + c;
eq0 = subs(L - (4-t^2), y, yp);
eq0 = collect(eq0, t)

eq0 =

(6*a+1)*t^2+(6*b+10*a)*t+2*a+6*c+5*b-4

[a b c] = solve(6*a+1, 6*b+10*a, 2*a+6*c+5*b-4)

a =

-1/6

b =

5/18

c =

53/108

%
yp = subs(yp); pretty(yp)

      2      53
      - 1/6 t  + 5/18 t + ---
                        108
check = subs(L, y, yp)

check =

4-t^2

%
y = yp + yh; pretty(y)

      2      53
      - 1/6 t  + 5/18 t + --- + c1 exp(-2 t) + c2 exp(-3 t)
                        108
sol = dsolve('D2y + 5*Dy + 6*y = 4 - t^2', 't');
pretty(sol)

      2      53
      C2 exp(-2 t) + exp(-3 t) C1 - 1/6 t  + 5/18 t + ---
                                                108

```

Example D [an initial value problem]

Solve the IVP $y'' + 2y' + 2y = 2 \cos 2t$, $y(0) = -2$, $y'(0) = 0$.

Solution

```

%
% NSS4-4.5/Example D
%
syms a b r t
y = sym('y(t)');
p = poly2sym([1 2 2], r); pretty(p)

      2
      r  + 2 r + 2
r = solve(p)

r =

[ -1+i]
[ -1-i]

yf = [exp(-t)*cos(t), exp(-t)*sin(t)];
%
L = diff(y,t,2) + 2*diff(y,t) + 2*y; pretty(L)

      / 2      \
      |d      |
      |--- y(t)| + 2 |d      |
      | 2      |    \dt      /
      \dt      /

yp = a*cos(2*t) + b*sin(2*t);
eq0 = subs(L - 2*cos(2*t), y, yp);
eq0 = collect(eq0, cos(2*t));
eq0 = collect(eq0, sin(2*t))

eq0 =

(-2*b-4*a)*sin(2*t)+(-2*a+4*b-2)*cos(2*t)

[a b] = solve(-2*b-4*a, -2*a+4*b-2)

a =

-1/5

b =

2/5

%
yp = subs(yp); pretty(yp)

      2/5 sin(2 t) - 1/5 cos(2 t)
check = subs(L, y, yp)

check =

2*cos(2*t)

%
v = [yf yp];
M = wron(v, t); % Push wron beyond its design specs!
M = simple(subs(M, t, sym(0)));
a = M(1:2, 3) % "Can you say subvectors and

a =

[ -1/5]
[ 4/5]

M = M(1:2, 1:2) % submatrices? I knew you could..."

M =

[ 1, 0]
[-1, 1]

b = [-2; 0]
b =

-2
0

```

```

c = M\b-a)

c =

[ -9/5]
[ -13/5]

y = yp + yf*c; pretty(y)

- 13/5 exp(-t) sin(t) - 9/5 exp(-t) cos(t)
- 1/5 cos(2 t) + 2/5 sin(2 t)
%
sol = dsolve('D2y + 2*Dy + 2*y = 2*cos(2*t)', ...
'y(0)=-2', 'Dy(0)=0', 't');
pretty(sol)

- 13/5 exp(-t) sin(t) - 9/5 exp(-t) cos(t)
- 1/5 cos(2 t) + 2/5 sin(2 t)
%
echo off; diary off

```

Example E [The forcing function is a solution of the homogeneous equation.]

Find a general solution of $y'' + 4y' + 4y = 2e^{-2t}$. Here the forcing term is also a solution of the associated homogeneous equation. Note the corresponding adjustment to the form of the particular solution!

Solution

```

%
% NSS4-4.5/Example E
%
syms a b c1 c2 r t
y = sym('y(t)');
p = poly2sym([1 4 4], r); pretty(p)

      2
      r  + 4 r + 4
r = solve(p)

r =

[ -2]
[ -2]

yh = c1*exp(-2*t) + c2*t*exp(-2*t);
%
L = diff(y,t,2) + 4*diff(y,t) + 4*y; pretty(L)

      / 2      \
      |d      |
      |--- y(t)| + 4 |--- y(t)| + 4 y(t)
      | 2      |
      \dt      /
yp = t^2 * a*exp(-2*t)

yp =

t^2*a*exp(-2*t)

eq0 = subs(L - 2*exp(-2*t), y, yp);
eq0 = collect(eq0, exp(-2*t)) / exp(-2*t)

eq0 =

2*a-2

a = solve(2*a-2)

a =

1

```

```

%
yp = subs(yp); pretty(yp)

      2
      t  exp(-2 t)
check = subs(L, y, yp)

check =

2*exp(-2*t)

y = yp + yh; pretty(y)

      2
      t  exp(-2 t) + c1 exp(-2 t) + c2 t exp(-2 t)
%
sol = dsolve('D2y + 4*Dy + 4*y = 2*exp(-2*t)', 't');
pretty(sol)

      exp(-2 t) C2 + t exp(-2 t) C1 + t      2
      exp(-2 t)
%
echo off; diary off

```

Example F [multiterm forcing function]

Find a general solution of $y'' + 16y = e^{-4t} + 3 \sin 4t$.

Your authors would have you split the forcing term up, handle each subproblem separately, then superimpose the solutions. That's fine, perhaps even preferable when doing things by hand. With massive firepower on tap, however, there is no need for this.

Note once again that a part of the forcing term is also a solution of the associated homogeneous equation. Accordingly, we make an adjustment in the form of that part of the particular solution.

Finally, with a little reassessment, we see that **dsolve**'s solution is equivalent to ours.

Solution

```

%
% NSS4-4.5/Example F
%
syms a b c c1 c2 r t C2 K
y = sym('y(t)');
p = poly2sym([1 0 16], r); pretty(p)

      2
      r  + 16
r = solve(p)

r =

[ 4*i]
[ -4*i]

yh = c1*cos(4*t) + c2*sin(4*t);
%
L = diff(y,t,2) + 16*y; pretty(L)

      / 2      \
      |d      |
      |--- y(t)| + 16 y(t)
      | 2      |
      \dt      /
yp = a*exp(-4*t) + t*(b*cos(4*t) + c*sin(4*t));
eq0 = subs(L - (exp(-4*t) + 3*sin(4*t)), y, yp);
eq0 = collect(eq0, exp(-4*t));
eq0 = collect(eq0, cos(4*t));
eq0 = collect(eq0, sin(4*t))

```

```

eq0 =
(-8*b-3)*sin(4*t)+8*c*cos(4*t)+(32*a-1)*exp(-4*t)

[a b c] = solve(-8*b-3, 8*c, 32*a-1)

a =
1/32

b =
-3/8

c =
0

%
yp = subs(yp); pretty(yp)

1/32 exp(-4 t) - 3/8 t cos(4 t)
check = subs(L, y, yp)

check =
exp(-4*t)+3*sin(4*t)

Y = yp + yh; pretty(Y)

1/32 exp(-4 t) - 3/8 t cos(4 t) + c1 cos(4 t) + c2 sin(4 t)
%
sol = dsolve('D2y + 16*y = exp(-4*t) + 3*sin(4*t)', 't');
sol = simple(sol); pretty(sol)

sin(4 t) C2 + cos(4 t) C1 - 3/8 t cos(4 t)
+ 3/32 sin(4 t) + 1/32 exp(-4 t)
sol = collect(sol, sin(4*t));
sol = subs(sol, C2, sym(K - 3/32));
pretty(sol) % It's equivalent to our solution!

K sin(4 t) + 1/32 exp(-4 t) + cos(4 t) C1 - 3/8 t cos(4 t)
%
echo off; diary off

```

```

yp = t * (a*t + b)*exp(-t);
eq0 = subs(L - t*exp(-t), y, yp);
eq0 = collect(eq0, exp(-t)) / exp(-t)

eq0 =
2*a+6*a*t+3*b-t

eq0 = collect(eq0, t)

eq0 =
(6*a-1)*t+2*a+3*b

[a b] = solve(6*a-1, 2*a+3*b)

a =
1/6

b =
-1/9

%
yp = subs(yp); pretty(yp)

t (1/6 t - 1/9) exp(-t)
check = subs(L, y, yp)

check =
1/3*exp(-t)+3*(1/6*t-1/9)*exp(-t)+1/2*t*exp(-t)

Y = yp + yh; pretty(Y)

t (1/6 t - 1/9) exp(-t) + c1 exp(-t) + c2 exp(-4 t)
%
sol = dsolve('D2y + 5*Dy + 4*y = t*exp(-t)', 't');
sol = simple(sol);
pretty(sol)

exp(-4 t) C2 + exp(-t) C1 + (1/6 t2 - 1/9 t) exp(-t)
%
echo off; diary off

```

Example G

Find a general solution of $y'' + 5y' + 4y = te^{-t}$. There are subtleties here too, but by now you're a pro...

```

%
% NSS4-4.5/Example G
%
syms a b c1 c2 r t
y = sym('y(t)');
p = poly2sym([1 5 4], r); pretty(p)
r2
r + 5 r + 4
r = solve(p)

r =
[ -1]
[ -4]

yh = c1*exp(-t) + c2*exp(-4*t);
%
L = diff(y,t,2) + 5*diff(y,t) + 4*y; pretty(L)

/ 2 \
| d | | /d \
|--- y(t)| + 5 |--- y(t)| + 4 y(t)
| 2 | | \dt /
\dt /

```