Fall 2003 Math 308/501–502
4 Linear Second-Order Equations
4.9 Forced Mechanical Vibrations
Fri, 03/Oct ©2003, Art Belmonte

Summary

When $x'' + cx' + \omega_0^2 x = A\cos\omega t$, we have a sinusoidal forcing term. The constant A is the **amplitude** of the driving force, whereas ω is the **driving frequency**. If $c = 0$, we have **forced undamped harmonic motion** whereas when $c > 0$, the we have **forced damped harmonic motion**.

Forced undamped harmonic motion

When the driving frequency ω is close but not equal to the natural frequency ω_0 of the system, a phenomenon called **beats** is exhibited. When $\omega = \omega_0$, a phenomenon called **resonance** is exhibited.

Forced damped harmonic motion

In this case the solution is consists of the sum of a **transient response** and a **steady-state repsonse**.

NOTE

There is more detail in this section than is contained in this summary. It is especially pertinent in electrical and electronic engineering. We shall, however, omit it here for brevity.

Hand Examples

Example C [beats]

Plot the function $\cos 9t - \cos 10t$ over $0 \le t \le 4\pi$. The phenomenon exhibited is called "beats." Use a trig identity to write the function as a product of sines. This shows that the function is a fast oscillation with a amplitude that oscillates more slowly. Superimpose the graph of the slower oscillating amplitude on the same plot using a different color and/or line style.

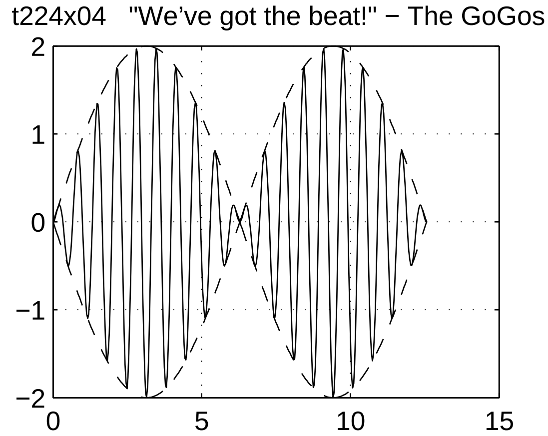
A piano tuner utilizes beats when tuning a piano. It is also used in amplitude modulation or AM radio broadcasting.

Solution

Recall the following trig identity.

$\cos\alpha - \cos\beta = -2\sin\frac{1}{2}(\alpha+\beta)\sin\frac{1}{2}(\alpha-\beta) = 2\sin\frac{\beta-\alpha}{2}\sin\frac{\beta+\alpha}{2}$

Therefore, $\cos 9t - \cos 10t = 2\sin(\frac{1}{2}t)\sin(\frac{19}{2}t)$. Here is the plot.



MATLAB Examples

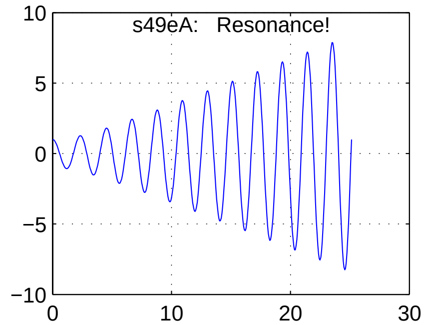
For review purposes, we'll use a variety of techniques to solve the initial value problems involved.

Example A: resonance

Determine an equation of motion for the forced undamped equation $y'' + 9y = 2\cos 3t$ that satisfies the initial conditions $y(0) = 1$, $y'(0) = 0$. Plot the solution. The phenomenon exhibited is called "resonance." It is why, for example, soldiers break ranks (i.e., do not march in step) when traversing a bridge. Resonance is also used in electrical systems to tune radios.

Solution

You can use the method of undetermined coefficients to obtain a particular solution. Note that the forcing term is a solution of the corresponding homogeneous equation. Hence the form of our particular solution is adjusted accordingly. Here we just use **dsolve**. A plot is followed by a MATLAB input file.



```
%
% Section 4.9 / Example A
% Forced undamped harmonic motion;
% example of resonance;
% solution via dsolve.
%
y = dsolve('D2y + 9*y = 2*cos(3*t)', ...
    'y(0)=1', 'Dy(0)=0', 't');
pretty(y)
%
t = linspace(0, 8*pi, 400);
y = eval(vectorize(y));
plot(t,y); grid on
%
echo off; diary off;
```

Example B: beats

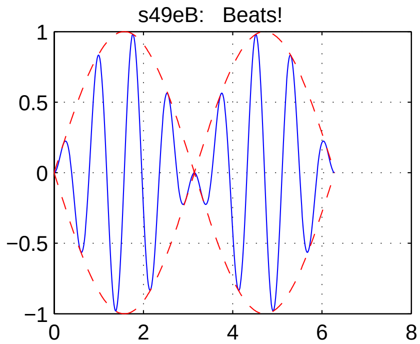
Solve the IVP $2y'' + 162y = 32\cos 7t$, $y(0) = 0$, $y'(0) = 0$. Plot the solution. The phenomenon exhibited is another example of beats.

Solution

You can use variation of parameters (Section 4.6) to obtain a particular solution. Note that we must first put the differential equation in standard linear form (SLF)!

$$y'' + 81y = 16\cos 7t$$

Here we just use **dsolve**. A plot is followed by a MATLAB input file.



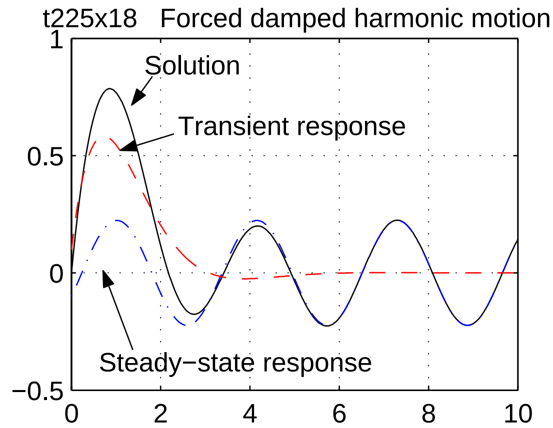

```
%
% Section 4.9 / Example B
% Forced undamped harmonic motion; example of beats;
% solution via dsolve.
%
y = dsolve('D2y + 81*y = 16*cos(7*t)', ...
    'y(0)=0', 'Dy(0)=0', 't');
pretty(y)
%
t = linspace(0, 2*pi, 400);
y1 = eval(vectorize(y));
y2 = sin(t);
plot(t,y1, t,y2,'r--', t,-y2,'r--'); grid on
%
echo off; diary off;
```

Example D [forced damped harmonic motion]

Solve the IVP $x'' + 2x' + 2x = \cos 2t$, $x(0) = 0$, $x'(0) = 2$. Plot the solution along with the transient and steady-state responses.

Solution

Here we dispatch the problem with **dsolve**. Here is a plot followed by a MATLAB diary file.




```
%
% NSS4-4.9/Example D
%
% Forced damped harmonic motion,
% illustrating transient and
% steady-state responses. Solution
% via dsolve.
%
sol = dsolve('D2x + 2*Dx + 2*x = cos(2*t)', ...
    'x(0)=0', 'Dx(0)=2', 't')

sol =

  17/10*exp(-t)*sin(t)+1/10*exp(-t)*cos(t)
+ 1/5*sin(2*t)-1/10*cos(2*t)

pretty(sol)

    17
    -- exp(-t) sin(t) + 1/10 exp(-t) cos(t)
    10
+ 1/5 sin(2 t) - 1/10 cos(2 t)
%
t = linspace(0, 10, 401);
x = eval(vectorize(sol));
ss = 1/5*sin(2*t) - 1/10*cos(2*t);
tr = 17/10*exp(-t) .* sin(t)  ...
   +  1/10*exp(-t) .* cos(t);
plot(t,x,'k', t,ss,'b-.', t,tr,'r--'); grid on
%
echo off; diary off;
```