Fall 2003 Math 308/501–502
6 Theory of Higher-Order Linear ODEs
6.1 Basic Theory of Linear ODEs
Wed, 24/Sep ©2003, Art Belmonte

Summary

We give an overview of the theory of nth order linear differential equations; here $n \geq 1$. Of course, the case where $n = 1$ was dealt with in Section 2.3, whereas $n = 2$ is dealt with in Chapter 4. Using summation and matrix notation along with a smattering of linear algebra concepts, we treat the general case once and for all. In the following, $I = (a, b)$ is an open real interval.

TERMINOLOGY

A **linear** ODE of order n has the form $\sum_{k=0}^{n} a_k(x) y^{(k)}(x) = b(x)$.

Here the a_k depend only on x (the independent variable), not on y (the dependent variable). The equation has **constant coefficients** if the a_k are constants; otherwise, it has **variable coefficients**. If $b(x)$ or $g(x)$ is zero on I, then the equation is **homogeneous**; otherwise it is **nonhomogeneous**.

With the a_k and b continuous on I and $a_n(x) \neq 0$ on I, divide to obtain the **standard form** $y^{(n)}(x) + \sum_{j=1}^{n} p_j(x) y^{(n-j)}(x) = g(x)$ or $L[y](x) = g(x)$, where $L = D^n + \sum_{j=1}^{n} p_j D^{n-j}$ is called a **linear differential operator**; i.e., $L\left[\sum_{i=1}^{m} y_i\right] = \sum_{i=1}^{m} L[y_i]$ and $L[cy] = cL[y]$. (This follows from properties of differentiation.)

DEFINITIONS

Let $f_1, \ldots, f_n$ be n functions that are differentiable $(n-1)$ times. The **Wronskian matrix** is $n \times n$ square array of derivatives

$$M = \begin{bmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{bmatrix},$$

Its determinant W is called the **Wronskian** [determinant].

The m functions $f_1, \ldots, f_m$ are **linearly dependent** on I if there exist constants $c_1, \ldots c_m$, not all zero, such that $\sum_{k=1}^{m} c_k f_k(x) = 0$ for all $x \in I$. Otherwise, the functions are **linearly independent**. (Note that in the linearly dependent case, one function is a linear combination of the other $m - 1$ functions.)

THEOREMS

Existence and Uniqueness Let $g(x)$ and $p_k(x)$ be continuous on I, an interval containing x_0. For any choice of constants γ_k, there exists a *unique* solution $y(x)$ on the *entire* interval I to the initial value problem

$$\begin{aligned} y^{(n)}(x) + \sum_{j=1}^{n} p_j(x) y^{(n-j)}(x) &= g(x), \\ y^{(k)}(x_0) &= \gamma_k, \quad k = 0, \ldots, n-1. \end{aligned}$$

Representation of Solutions (Homogeneous Case) Suppose that $y_1, \ldots, y_n$ are n solutions on I of

$$y^{(n)}(x) + \sum_{j=1}^{n} p_j(x) y^{(n-j)}(x) = 0, \qquad (*)$$

where the p_k are continuous on I. If the Wronskian of the y_k is nonzero at *some* point $x_0 \in I$, then *every* solution of (*) on I may be expressed as $y(x) = \sum_{k=1}^{n} c_k y_k(x)$, where the c_k are constants. This is the **general solution** of the homogeneous equation.

Linear (in)dependence and the Wronskian Let $y_1, \ldots, y_n$ be n solutions on I of $y^{(n)}(x) + \sum_{j=1}^{n} p_j(x) y^{(n-j)}(x) = 0$, with the p_k continuous on I. Then these three statements are equivalent:

- The y_k are linearly dependent on I.
- The Wronskian W of the y_k is zero at some point $x_0 \in I$.
- The Wronskian W of the y_k is identically zero on I.

The following are also equivalent. If one is true, then $\{y_1, \ldots, y_n\}$ is a **fundamental solution set** of the homogeneous equation on I.

- The y_k are linearly independent on I.
- The Wronskian W of the y_k is nonzero at some point $x_0 \in I$.
- The Wronskian W of the y_k is never zero on I.

Representation of Solutions (Nonhomogeneous Case) Let $y_p(x)$ be a particular solution of the nonhomogeneous equation $L[y](x) = g(x)$ on I, and let $\{y_1, \ldots, y_n\}$ be a fundamental solution set for the corresponding homogeneous equation $L[y](x) = 0$. Then every solution on I of $L[y](x) = g(x)$ may be expressed as $y = y_p + \sum_{k=1}^{n} c_k y_k$, where the $c_k$are constants. This is the **general solution** of the nonhomogeneous equation.

Hand Examples

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Determine the largest interval (a, b) for which the Existence and Uniqueness Theorem (EUT) guarantees the existence of a unique solution on (a, b) to the initial value problem

$$x(x+1)y''' - 3xy' + y = 0;\ \ y(-\tfrac{1}{2}) = 1,\ \ y'(-\tfrac{1}{2}) = y''(-\tfrac{1}{2}) = 0.$$

Solution

First put the differential equation in standard linear form:

$$y''' - \frac{3}{x+1}y' + \frac{1}{x(x+1)}y = 0.$$

Thus $p_1(x) = 0$, $p_2(x) = -\dfrac{3}{x+1}$, $p_3(x) = \dfrac{1}{x(x+1)}$, and $g(x) = 0$. These four functions are simultaneously continuous on $(a, b) = (-1, 0)$. This is the largest open interval containing $x_0 = -\frac{1}{2}$ on which they are continuous and thus satisfy the conditions of the EUT.

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Determine whether the functions $\left\{x^2, x^2 - 1, 5\right\}$ are linearly independent on $(-\infty, \infty)$.

Solution

Observe that $(5)\left(x^2\right) + (-5)\left(x^2 - 1\right) + (-1)(5) = 0$ on $(-\infty, \infty)$. Therefore the stated functions are linearly *dependent* on $(-\infty, \infty)$ by definition.

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Determine whether the functions $\left\{x, xe^x, 1\right\}$ are linearly independent on $\mathbb{R} = (-\infty, \infty)$.

Solution

Assume that the functions are linearly dependent. Then there exist constants c_1, c_2, c_3, *not all zero*, such that

$$c_1x + c_2xe^x + c_3 \cdot 1 = 0 \text{ for all } x \in \mathbb{R}.$$

- In particular, this is true for $x = 0$, whence $c_3 = 0$. Thus $c_1x + c_2xe^x = 0$ for all $x \in \mathbb{R}$.
- Differentiating this last equation with respect to x gives $c_1 + c_2(x+1)e^x = 0$ for all $x \in \mathbb{R}$. In particular, this is true for $x = -1$, whence $c_1 = 0$. So $c_2(x+1)e^x = 0$ for all $x \in \mathbb{R}$.
- In particular, this last equation is true for $x = 0$. This gives $c_2 = 0$.
- Therefore, c_1, c_2, c_3 are all zero, a contradiction.

Accordingly, the assumption that the functions were linearly dependent is false. Therefore, they are linearly independent.

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Let $L[y] = y''' + y' + xy$, $y_1(x) = \sin x$, and $y_2(x) = x$. Verify that $L[y_1](x) = x \sin x$ and $L[y_2](x) = x^2 + 1$. Then use the superposition principle (the fact that L is a *linear* operator) to find a solution to these differential equations.

- $L[y](x) = 2x \sin x - x^2 - 1$
- $L[y](x) = 4x^2 + 4 - 6x \sin x$

Solution

- See the MATLAB examples for verification that $L[y_1](x) = x \sin x$ and $L[y_2](x) = x^2 + 1$.
- Let $y = 2y_1 - y_2$. Then

$$\begin{aligned} L[y](x) &= L[2y_1 - y_2](x) \\ &= 2L[y_1](x) - L[y_2](x) \\ &= 2x \sin x - x^2 - 1. \end{aligned}$$

  Thus $y(x) = 2y_1(x) - y_2(x) = 2 \sin x - x$ solves the DE.

- Let $y = 4y_2 - 6y_1$. Then

$$\begin{aligned} L[y](x) &= L[4y_2 - 6y_1](x) \\ &= 4L[y_2](x) - 6L[y_1](x) \\ &= 4x^2 + 4 - 6x \sin x. \end{aligned}$$

  Thus $y(x) = 4y_2(x) - 6y_1(x) = 4x - 6 \sin x$ solves the DE.

MATLAB Examples

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Using the Wronskian, verify that the functions $\left\{e^x, \cos 2x, \sin 2x\right\}$ form a fundamental solution set for the differential equation

$$y''' - y'' + 4y' - 4y = 0.$$

Then find a general solution.

Solution

We verify via direct substitution that the three functions specified are indeed solutions of the differential equation. Next, their Wronskian is seen to be nonzero on $\mathbb{R}$, so the three functions are linearly independent. Accordingly, they form a fundamental solution set for the given 3rd-order differential equation.
A general solution is $y(x) = c_1 e^x + c_2 \cos 2x + c_3 \sin 2x$.

Here is the short script M-file.

```
%
% NSS4-324/16
%
syms x
y = sym('y(x)')
de0 = diff(y,x,3) - diff(y,x,2) + 4*diff(y,x) - 4*y;
pretty(de0)
%
y1 = exp(x); y2 = cos(2*x); y3 = sin(2*x);
check1 = subs(de0, y, y1)
check2 = subs(de0, y, y2)
check3 = subs(de0, y, y3)
%
v = [y1 y2 y3]
M = wron(v, x)
W = simple(det(M))
%
echo off; diary off
```

Here's the corresponding diary file with input and output interspersed.

```
%
% NSS4-324/16
%
syms x
y = sym('y(x)')

y =

y(x)

de0 = diff(y,x,3) - diff(y,x,2) + 4*diff(y,x) - 4*y;
pretty(de0)

   / 3       \   / 2       \
   |d        |   |d        |     /d      \
   |--- y(x)| - |--- y(x)| + 4 |-- y(x)| - 4 y(x)
   |  3      |   |  2      |     \dx     /
   \dx       /   \dx       /
%
y1 = exp(x); y2 = cos(2*x); y3 = sin(2*x);
check1 = subs(de0, y, y1)

check1 =

0

check2 = subs(de0, y, y2)

check2 =

0

check3 = subs(de0, y, y3)

check3 =

0

%
v = [y1 y2 y3]

v =

[   exp(x), cos(2*x), sin(2*x)]

M = wron(v, x)

M =

[      exp(x),    cos(2*x),    sin(2*x)]
[      exp(x), -2*sin(2*x),  2*cos(2*x)]
[      exp(x), -4*cos(2*x), -4*sin(2*x)]

W = simple(det(M))

W =

10*exp(x)

%
echo off; diary off
```

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The function $y_p = \cos x$ is a particular solution of the nonhomogeneous DE $y^{(4)} + 4y = 5\cos x$. A fundamental solution set for the corresponding homogeneous equation $y^{(4)} + 4y = 0$ is $\{y_1, y_2, y_3, y_4\} = \left\{e^x \cos x, e^x \sin x, e^{-x} \cos x, e^{-x} \sin x\right\}$.

- (a) Find a general solution to the nonhomogeneous equation.
- (b) Find the solution that satisfies the initial conditions

$$y(0) = 2,\ y'(0) = 1,\ y''(0) = -1,\ y'''(0) = -2.$$

Solution

- (a) A general solution to the nonhomogeneous equation is given by $y = y_p + c_1 y_1 + c_2 y_2 + c_3 y_3 + c_4 y_4$.

$$\begin{aligned} y(x) &= \cos x + c_1 e^x \cos x + c_2 e^x \sin x \\ &\quad + c_3 e^{-x} \cos x + c_4 e^{-x} \sin x \end{aligned}$$

- (b) Now solve for the c_k the MATLAB way! (What does he mean by that?) Let $\mathbf{y}_f = [y_1, y_2, y_3, y_4]$ be a row vector that constitutes the fundamental solution set. Define a *column* vector of constants we are trying to find: $\mathbf{c} = [c_1; c_2; c_3; c_4]$. Recall the matrix/vector notation from chapters 2 & 11 of your lab manual. (You *are* working through those problems, yes?!) In general, we have $y = y_p + \mathbf{y}_f \cdot \mathbf{c}$, whence, via repeated differentiation,
$y^{(k)} = y_p^{(k)} + \mathbf{y}_f^{(k)} \cdot \mathbf{c},\ k = 0, 1, \ldots, n-1$.

  The ICs give $\gamma_k = y^{(k)}(x_0) = y_p^{(k)}(x_0) + \mathbf{y}_f^{(k)}(x_0) \cdot \mathbf{c}$ in general. Accordingly, let $\mathbf{b} = [\gamma_0; \gamma_1; \ldots; \gamma_{n-1}]$ and $\mathbf{a} = [y_p(x_0); y_p'(x_0); \ldots; y_p^{(n-1)}(x_0)]$; note that these are *column* vectors. Moreover, let $\mathbf{M}$ be the Wronskian matrix of $\mathbf{y}_f$ (the fundamental set of solutions) evaluated at x_0. Then we have $\mathbf{b} = \mathbf{a} + \mathbf{M}\mathbf{c}$, whence $\mathbf{c} = \mathbf{M}^{-1}(\mathbf{b} - \mathbf{a})$, which is realized in MATLAB as $\mathbf{c} = \mathbf{M}\backslash(\mathbf{b} - \mathbf{a})$. (Yes, Studenten, that's *left* division, not right division. There *is* a difference between the two when matrices are involved. But you know this from chapters 2 & 11 of your lab manual, right?)

So let's run it up the flagpole and see if anyone salutes. Indeed, they do! We see that the unique solution to the IVP is $y(x) = \cos x + e^x \cos x$. First look at the script M-file.

```
%
delete p325x22.txt; diary p325x22.txt
clear; clc; close all; echo on
%
% NSS4-325/22
%
format rat
syms x
yf = [exp(x)*cos(x), exp(x)*sin(x), ... % Fund soln set
        exp(-x)*cos(x), exp(-x)*sin(x)];
yp = cos(x); % Particular soln
v = [yf yp]; % Push wron beyond its design specs...
M = wron(v, x); a = M(1:4, 5); M = M(1:4, 1:4);
a = subs(a, x, 0); % yp at x0
M = subs(M, x, 0)  % Wronskian matrix at x0
b = [2; 1; -1; -2] % the ICs
c = M\(b-a)        % MAGIC!
y = yp + yf*c; pretty(y) % Put the pieces together.
%
sol = dsolve('D4y + 4*y = 5*cos(x)', 'y(0)=2', ...
    'Dy(0)=1', 'D2y(0)=-1', 'D3y(0)=-2', 'x');
pretty(sol) % dsolve agrees!
%

echo off; diary off
```

Now examine the diary file.

```
%
% NSS4-325/22
%
format rat
syms x
yf = [exp(x)*cos(x), exp(x)*sin(x), ... % Fund soln set
        exp(-x)*cos(x), exp(-x)*sin(x)];
yp = cos(x); % Particular soln
v = [yf yp]; % Push wron beyond its design specs...
M = wron(v, x); a = M(1:4, 5); M = M(1:4, 1:4);
a = subs(a, x, 0); % yp at x0
M = subs(M, x, 0)  % Wronskian matrix at x0
M =
      1            0            1            0
      1            1           -1            1
      0            2            0           -2
     -2            2            2            2
b = [2; 1; -1; -2] % the ICs
b =
      2
      1
     -1
     -2
c = M\(b-a)        % MAGIC!
c =
      1
      0
      0
      0
y = yp + yf*c; pretty(y) % Put the pieces together.

                            cos(x) + exp(x) cos(x)
%
sol = dsolve('D4y + 4*y = 5*cos(x)', 'y(0)=2', ...
    'Dy(0)=1', 'D2y(0)=-1', 'D3y(0)=-2', 'x');
pretty(sol) % dsolve agrees!

                            cos(x) + exp(x) cos(x)
```

325/23 [revisited]

Here are the promised verifications.

```
%
% NSS4-325/23
%
syms x
y = sym('y(x)');
y1 = sin(x); y2 = x;
L = diff(y,x,3) + diff(y,x) + x*y;
pretty(L)

                        / 3        \
                        |d         |   /d        \
                        |--- y(x)| + |-- y(x)| + x y(x)
                        |  3       |   \dx       /
                        \dx        /
check1 = subs(L, y, y1)

check1 =

x*sin(x)

check2 = subs(L, y, y2)

check2 =

1+x^2

%

echo off; diary off
```