

Fall 2003 Math 308/501–502
6 Theory of Higher-Order Linear ODEs
6.2 Homogeneous Linear Equations
with Constant Coefficients
 Fri, 26/Sep ©2003, Art Belmonte

Summary

A homogeneous linear differential equation of order n with real constant coefficients has the form $\sum_{k=0}^n a_k y^{(k)}(x) = 0$. Its

associated **auxiliary** or **characteristic equation** is $\sum_{k=0}^n a_k r^k = 0$.

By the Fundamental Theorem of Algebra, there are n roots of this polynomial equation (accounting for multiplicities), which are real or occur in complex conjugate pairs. Here is how to find solutions to the differential equation.

- Let r be a real root multiplicity m . Then $x^k e^{rx}$, $k = 0, 1, \dots, m - 1$, are m linearly independent solutions of the DE.
- Let $r = \alpha + i\beta$ be a complex root of the characteristic polynomial with multiplicity m . Then we have the following $2m$ linearly independent solutions of the DE.

$$x^k e^{\alpha x} \cos \beta x, \quad k = 0, 1, \dots, m - 1;$$

$$x^k e^{\alpha x} \sin \beta x, \quad k = 0, 1, \dots, m - 1.$$

“Hand” Examples

When necessary, resort to MATLAB for computations.

Example A

Show that $y_1(x) = e^x$, $y_2(x) = xe^x$, and $y_3(x) = x^2e^x$ are linearly independent.

Solution

Assume these functions are linearly dependent. Then there exist constants c_1, c_2, c_3 , not all zero, such that $c_1 y_1 + c_2 y_2 + c_3 y_3 = 0$ for all x . In particular, for $x = -1, 0, 1$, we have

$$\begin{bmatrix} e^{-1} & -e^{-1} & e^{-1} \\ 1 & 0 & 0 \\ e & e & e \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or $\mathbf{M}\mathbf{c} = \mathbf{0}$, whence $\mathbf{c} = \mathbf{M}\mathbf{0} = [0; 0; 0]$, a contradiction (since not all the c_k are zero). Hence it must be the case that all the c_k are zero. Thus y_1, y_2 , and y_3 are linearly independent. (NOTE: This is a much quicker way of showing linear independence directly that what we did in the Section 6.1 lecture handout.)

Example B

Show that $y_1(x) = e^{3x}$, $y_2(x) = \cos 3x$, $y_3(x) = \sin 3x$ form a fundamental set of solutions of the third-order equation $y''' - 3y'' + 9y' - 27y = 0$.

Solution

- Via direct substitution,

$$y_1''' - 3y_1'' + 9y_1' - 27y_1 = 27e^{3x} - 3(9e^{3x}) + 9(3e^{3x}) - 27(e^{3x}) = 0.$$

In similar manner, we have that $y_2''' - 3y_2'' + 9y_2' - 27y_2 = 0$ and $y_3''' - 3y_3'' + 9y_3' - 27y_3 = 0$.

- The Wronskian determinant of y_1, y_2, y_3 is

$$W(x) = \det \begin{bmatrix} e^{3x} & \cos 3x & \sin 3x \\ 3e^{3x} & -3 \sin 3x & 3 \cos 3x \\ 9e^{3x} & -9 \cos 3x & -9 \sin 3x \end{bmatrix}.$$

$$\text{At } x = 0, \text{ we have } W(x) = \det \begin{bmatrix} 1 & 1 & 0 \\ 3 & 0 & 3 \\ 9 & -9 & 0 \end{bmatrix} = 54 \neq 0.$$

Thus y_1, y_2, y_3 are linearly independent.

Example C

Find a general solution of

$$y^{(5)} - 4y^{(4)} - 13y''' + 52y'' + 36y' - 144y = 0.$$

Solution

The characteristic equation

$$r^5 - 4r^4 - 13r^3 + 52r^2 + 36r - 144 = 0$$

has distinct roots $r = -3, -2, 2, 3, 4$, whence

$$y = c_1 e^{-3x} + c_2 e^{-2x} + c_3 e^{2x} + c_4 e^{3x} + c_5 e^{4x}.$$

Example D

Find a general solution of

$$y^{(5)} - y^{(4)} - 6y''' + 14y'' - 11y' + 3y = 0.$$

Solution

The characteristic equation

$$r^5 - r^4 - 6r^3 + 14r^2 - 11r + 3 = 0$$

has roots $\lambda = -3, 1, 1, 1, 1$, whence

$$y = c_1 e^{-3x} + c_2 e^x + c_3 x e^x + c_4 x^2 e^x + c_5 x^3 e^x.$$

Example E

Find a general solution of

$$y^{(6)} + 3y^{(4)} + 3y'' + y = 0.$$

Solution

The characteristic equation

$$r^6 + 3r^4 + 3r^2 + 1 = 0$$

has roots $\lambda = i, i, i, -i, -i, -i$, whence

$$y = c_1 \cos x + c_2 \sin x + c_3 x \cos x + c_4 x \sin x + c_5 x^2 \cos x + c_6 x^2 \sin x.$$

Example F

Solve the initial value problem

$$y'' - 2y' + y = 0, \quad y(0) = 1, \quad y'(0) = 0.$$

Solution

The characteristic equation $r^2 - 2r + 1 = 0$ has roots $r = 1, 1$, whence $y = c_1 e^x + c_2 x e^x$ and $y' = c_1 e^x + c_2(x + 1)e^x$. Thus

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

whence $c_1 = 1$ and $c_2 = -1$. Therefore, $y = e^x - x e^x$. (NOTE: The numerical coefficient matrix above is the Wronskian matrix of $y_1 = e^x$ and $y_2 = x e^x$ evaluated at $x = 0$. This is used to great advantage in the MATLAB examples below!)

MATLAB Examples

Example B [revisited]

In the usual manner, we find the roots of the characteristic equation, construct our fundamental solution set, then establish linear independence.

```

%
% NSS4-6.2/Example B
%
syms r x
p = poly2sym([1 -3 9 -27], r); pretty(p)
                                     3      2
                                     r  - 3 r  + 9 r - 27

r = solve(p, r)

r =

[ 3]
[ 3*i]
[-3*i]

y1 = exp(3*x); y2 = cos(3*x); y3 = sin(3*x);
v = [y1 y2 y3]; M = subs(wron(v,x), x, 0)
M =
     1     1     0
     3     0     3
     9    -9     0
det(M)
ans =
     54
%
echo off; diary off

```

Example C [revisited]

Reload!

```

%
% NSS4-6.2/Example C
%
syms r x
p = poly2sym([1 -4 -13 52 36 -144], r); pretty(p)
                                     5      4      3      2
                                     r  - 4 r  - 13 r  + 52 r  + 36 r - 144

r = solve(p, r)

r =

[-2]
[ 2]
[-3]
[ 3]
[ 4]

%
echo off; diary off

```

Example E [revisited]

Make sure that you include ALL coefficients of the polynomial, including zeros!

```

%
% NSS4-6.2/Example E
%
syms r x
p = poly2sym([1 0 3 0 3 0 1], r); pretty(p)
                                     6      4      2
                                     r  + 3 r  + 3 r  + 1

r = solve(p, r)

r =

[-i]
[ i]
[-i] % (continued on next page)

```

```
[ i]
[ -i]
[ i]

%
echo off; diary off
```

Example G

Solve the initial value problem

$$y''' - 7y'' + 11y' - 5y = 0, \quad y(0) = -1, \quad y'(0) = 1, \quad y''(0) = 0.$$

Solution

We do the same steps as in Example F, only with MATLAB.

```
%
% NSS4-6.2/Example F
%
syms r t
p = poly2sym([1 -7 11 -5], r); pretty(p)

          3      2
r  - 7 r  + 11 r - 5

r = solve(p, r)

r =

[ 5]
[ 1]
[ 1]

y1 = exp(5*x); y2 = exp(x); y3 = x*exp(x);
yf = [y1 y2 y3];
%
W = simple(wron(yf,x))
%
%
W =

[ exp(5*x), exp(x), x*exp(x) ]
[ 5*exp(5*x), exp(x), exp(x)*(1+x) ]
[ 25*exp(5*x), exp(x), exp(x)*(2+x) ]

M = subs(W, x, 0)
M =

    1    1    0
    5    1    1
   25    1    2
b = sym([-1; 1; 0])

b =

[ -1]
[  1]
[  0]

c = M\b

c =

[ -3/16]
[ -13/16]
[  11/4 ]

y = yf*c; pretty(y)

      13
- 3/16 exp(5 x) - -- exp(x) + 11/4 t exp(x)
      16

%
echo off; diary off
```

Example H

Solve the initial value problem

$$y''' - 3y' + 52y = 0, \quad y(0) = 0, \quad y'(0) = -1, \quad y''(0) = 2.$$

Solution

```
%
% NSS4-6.2/Example G
%
syms r t
p = poly2sym([1 0 -3 52], r); pretty(p)

          3
r  - 3 r + 52

r = solve(p, r)

r =

[ -4]
[ 2+3*i]
[ 2-3*i]

y1 = exp(-4*x); y2 = exp(2*x)*cos(3*x);
y3 = exp(2*x)*sin(3*x);
yf = [y1 y2 y3];
%
W = simple(wron(yf,x)); % long and messy
M = subs(W, x, 0) % short and sweet!
M =

    1    1    0
   -4    2    3
   16   -5   12
b = sym([0; -1; 2])

b =

[  0]
[ -1]
[  2]

c = M\b

c =

[  2/15]
[ -2/15]
[ -1/15]

y = yf*c; pretty(y)

      2/15 exp(-4 x) - 2/15 exp(2 x) cos(3 x)
                    - 1/15 exp(2 x) sin(3 x)

%
echo off; diary off
```

332/31b

Use the substitution $y = x^r$ to solve the Cauchy Euler equation

$$x^4 y^{(4)} + 6x^3 y''' + 2x^2 y'' - 4xy' + 4y = 0.$$

Solution

Direct substitution, simplification, and solution gives $r = -2, -1, 1, 2$. Therefore, a general solution is $y(x) = c_1 x^{-2} + c_2 x^{-1} + c_3 x + c_4 x^2$. (See MATLAB diary file on next page.)

```

%
% NSS4-332/31b
%
syms r x
Y = sym('y(x)')

Y =

Y(x)

de0 = x^4*diff(y,x,4) + 6*x^3*diff(y,x,3) ...
      + 2*x^2*diff(y,x,2) - 4*x*diff(y,x) + 4*y;
pretty(de0)

      / 4      \
      |d      |
      |--- y(x)|
      | 4      |
      |dx      |
      | 3      |
      |d      |
      |--- y(x)|
      | 3      |
      |dx      |
      | 2      |
      |d      |
      |--- y(x)|
      | 2      |
      |dx      |
      | 1      |
      |d      |
      |--- y(x)|
      | 1      |
      |dx      |
      + 4 y(x)

%
eq0 = subs(de0, y, x^r);
eq0 = simple(eq0)

eq0 =

x^r*(r^4-5*r^2+4)

eq0 = eq0 / x^r

eq0 =

r^4-5*r^2+4

r = solve(eq0, r)

r =

[ -1]
[  1]
[ -2]
[  2]

%
echo off; diary off

```